Differ Equ Dyn Syst (Jan&Apr 2013) 21(1&2):35–43
DOI 10.1007/s12591-012-0121-y

ORIGINAL RESEARCH

# Occurrence of Big Bang Bifurcations in Discretized Sliding-mode Control Systems

**Enric Fossas · Albert Granados**

Received: 29 February 2012 / Accepted: 15 May 2012 / Published online: 29 May 2012


**Abstract** In this work we describe the bifurcation scenario found in a general first order system when a relay based proportional control (sliding-mode control) is considered. Based on the results given in the literature, we show the occurrence of a big bang bifurcation causing the existence of an infinite number of periodic orbits near a co-dimension two bifurcation point. We also extend in a natural way the applied theoretical result for second order systems involving 2D piecewise-defined maps.

**Keywords** Big bang bifurcations · Two-dimensional piecewise-defined maps · Period adding · Sliding-mode control · Relays

## Introduction

There exist many methods in order to force a system to exhibit a certain desired behaviour. If its output is required to be near a certain value, one common strategy consists on implementing a control system such that two different actions are applied depending on the sign of a certain switching function which depends on the actual state and its derivatives. In general, this leads to a non-smooth system which, among other phenomena can exhibit sliding.

As, in practice, the states are sampled at particular values of time, one considers a discretization of such a construction through a zero order holder, keeping the sampled value constant until the next sampling. Instead of a differential equation, the system is then usually modeled by a map, whose dynamics may differ completely from the time continuous system where a "continuous sampling" of the states (infinite sampling frequency) is assumed. This especially occurs when the states are close to the switching manifold, as this map does not

E. Fossas
Universitat Politècnica de Catalunya, IOC, Barcelona, Spain
e-mail: enric.fossas@upc.edu

A. Granados (✉)
Universität Stuttgart, IPVS, Stuttgart, Germany
e-mail: albert.granados@ipvs.uni-stuttgart.de

coincide with the stroboscopic Poincaré map of the time-continuous system. In particular, new bifurcation phenomena may be introduced.

Because of the nowadays hegemony of digital implementations, this has become a relevant topic in the control literature. In [19,10,18,5,11] the discretization effects of a sliding mode controller in a planar system is studied. Specifically, the discontinuous control results from the addition of the equivalent control plus a sign function, properly weighted. Then, a zero-order holder device is applied. As it is proven in [11], the resulting dynamics show an infinite number of periodic orbits with arbitrarily large periods near a certain point in a 2D space. Similar phenomenon were also shown in [13] for 1D systems derived from power converters.

Such points in parameter space assemble the so-called *big bang* bifurcations, first introduced in [2] when simulating a 1D piecewise-linear system, better understood and generalized later in [1]. Unfortunately, the theory derived so far only considers 1D maps and, hence, it can't be applied to the above mentioned planar systems in sliding-mode control.

It is worth mentioning here that, when controllers are implemented through switches, as in the case of power electronics, the resulting sliding-mode control actions reduce to the discontinuous term; *i.e.* $\varepsilon\text{sign}(\sigma)$. Then, the digitize dynamics matches perfectly with the maps that yield to *big bang* bifurcations. On the contrary, when the continuous term (the equivalent control) is included, the derived map is not longer contractive, which is highly required in the theoretical results obtained so far.

In this work, we use recent results for big bang bifurcations to explain the behaviour of a class of digitized sliding mode controlled first order systems. Specifically, in terms of [2], a big bang bifurcation of the period adding type is shown to happen in that systems when the on-off control is digitized. Moreover, big bang bifurcations are shown to happen in on-off sliding mode controlled planar systems. Since the theory is not complete in this case, sufficient conditions for such a bifurcation to occur in 2D piecewise maps are conjectured and corroborated by simulation.

## A System With a Relay Based Control

### System Description

Let us consider a $n$th-order system given by its Laplace transform

$$G_s(s) = \frac{b}{U(s)},$$

where $U(s)$ is a polynomial of the form

$$U(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0, \tag{1}$$

which we assume to have only negative real roots.

Equivalently, one can also consider that the system is modeled by the differential equation

$$y^{n)} + a_{n-1}y^{n-1)} + \ldots + a_0 y = bu,$$

where $y^{i)} = d^i y/dt^i$ and $u$ is the input of the system.

Let us suppose that we wish to control the system and make its output, $y$, close to a certain desired value $y_c$. Although there exist many ways to achieve this, we consider the closed loop control scheme shown in Fig. 1.

On one hand, a certain control law is implemented by the block $G_c(s)$, which is assumed to be of the form

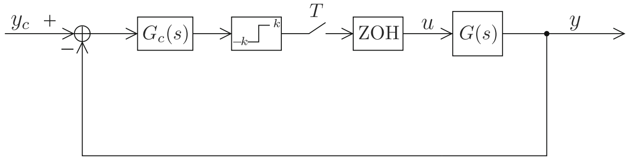


**Fig. 1** A linear control system by a relay

$$G_c(s) = 1 + c_1 s + \ldots + c_{n-1} s^{n-1},$$

with $c_{n-1} \neq 0$. This is to let the system have relative degree 1 and allow sliding motion on $\sigma$ (see below).

Its output is then sent to a relay of gain $k$, hence providing a sliding-mode control with a sliding surface (also called switching surface) given by the controller $G_c$,

$$\sigma := y - y_c + c_1(y^{1)} - y_c^{1)}) + \ldots + c_{n-1}(y^{n-1)} - y_c^{n-1)}) = 0. \tag{2}$$

In this work we consider $y_c \in \mathbb{R}$ constant, and thus $y_c^{i)} = 0$ for $i \geq 1$.

Depending on the sign of the signal given by the controller $G_c$, the relay outputs the value $k$ or $-k$, which yield sliding motions on $\sigma$ provided that the sign and the absolute value of $k$ are properly chosen.

Finally, the control output is digitized through a zero-order holder device, as in a real implementation. This is represented in Fig. 1 by a switch that samples the output of the relay at time-multiples of the sampling-period $T$ and a zero order holder, which keeps the sampled value constant until the next sampling. Close to the sliding surface (2), the dynamics of the discretized system differs from the time continuous one, although they tend to be the same as $T \to 0$. It is our goal to study the dynamics of the discretized system.

After performing a proper change of variables to decrease the order of the system by increasing its dimension, $y_i = y^{i)}$, the closed loop dynamics can be written as

$$\dot{\bar{y}} = A\bar{y} + \bar{b}u \tag{3}$$

with $\bar{y} = (y_0, \ldots, y_{n-1})^T \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ -a_0 & -a_1 & \ldots & & -a_{n-2} & -a_{n-1} \end{pmatrix}, \; \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \end{pmatrix}.$$

for $t \in [iT, (i+1)T)$, the input $u$ is a constant equal to

$$u = \begin{cases} -k & \text{if } \sigma(\bar{y}) < 0 \\ k & \text{if } \sigma(\bar{y}) > 0 \end{cases} \tag{4}$$

where $\sigma$ is the sliding surface given by the controller $G_c(s)$ in Eq. (2).

Sliding modes occur if the vector fields $F^{\pm} = A\,x \pm b\,k$ obtained by replacing $u = \pm k$, point both to the surface $\sigma$. Since $F^{\pm}$ are smooth everywhere, this can be checked through

$$(L_{F^+}\sigma)\,(L_{F^-}\sigma) < 0. \tag{5}$$

Let us define $u_{eq} = -\frac{(\nabla\sigma)A\bar{y}}{c_{n-1}b}$, then the previous inequality meets on the subset of $\sigma$ defined by

$$-|k| < u_{eq} < |k| \tag{6}$$

(see [17] for details). In turn, this result can be read as *for k properly selected (both in sign and in absolute value), there is sliding motion on $\sigma$.*

Equivalently, the dynamics of the system are given by the discrete model

$$\bar{y}_{i+1} = P(\bar{y}_i),$$

where $P_l$ (resp. $P_r$) is the piecewise defined stroboscopic map associated with $F^+$, (resp. $F^-$) which is linear and can be explicitly integrated. We obtain

$$P(\bar{y}) = \begin{cases} P_l(\bar{y}) := \bar{\rho}\bar{y} + \bar{\mu}_\ell & \text{if } \sigma(\bar{y}) < 0 \\ P_r(\bar{y}) := \bar{\rho}\bar{y} + \bar{\mu}_r & \text{if } \sigma(\bar{y}) > 0, \end{cases} \tag{7}$$

with

$$\bar{\rho} = e^{AT}, \quad \bar{\mu}_r = k(\bar{\rho} - Id)(A^{-1}\bar{b}), \quad \bar{\mu}_\ell = -k(\bar{\rho} - Id)(A^{-1}\bar{b}).$$

General system dynamics

Each branch of the map (9), $P_r$ and $P_\ell$, has a fixed point

$$\bar{y}_r^* = -(\bar{\rho} - Id)^{-1}\bar{\mu}_r, \quad \bar{y}_\ell^* = -(\bar{\rho} - Id)^{-1}\bar{\mu}_\ell, \tag{8}$$

which may be *feasible* or *virtual* depending on whether it belongs to the domain of their respective map or not.

Regarding the possible dynamics, we distinguish between three situations.

If both fixed points are feasible ($\sigma(\bar{y}_r^*) > 0$ and $\sigma(\bar{y}_\ell^*) < 0$) they also become fixed points of the map (9). Hence, if all eigenvalues of $\bar{\rho}$ have modulus less than 1, both are locally asymptotically stable. If only one of both fixed points is feasible ($\sigma(\bar{y}_r^*) < 0$ and $\sigma(\bar{y}_\ell^*) < 0$ or vice-versa) and the same condition for $\bar{\rho}$ holds, then it becomes the unique fixed point of the map (9). For the same reason, all trajectories tend towards it, and now its domain of attraction becomes $\mathbb{R}^n$.

Note that, in these two previous cases, the control specification is not fulfilled as $\sigma$ is not flow invariant by the piecewise vector field $F$.

The third situation occurs when both fixed points are virtual ($\sigma(\bar{y}_r^*) < 0$ and $\sigma(\bar{y}_\ell^*) > 0$). If all the eigenvalues of $\bar{\rho}$ have modulus <1 and are real (hence positive) then there are sliding motions in the original continuous-time system. In this case, and at least for linear and planar systems, the dynamics of the digitized map consists on periodic orbits which may possess arbitrarily large periods and whose iterates jump on both sides of the sliding surface $\sigma = 0$. Properly tunning the parameters, the amplitude of all these orbits can be chosen arbitrarily small. Additionally, the design conditions are satisfied in this case, as the asymptotic dynamics are close to $\sigma = 0$. A precise description of all the possible periodic orbits is the main scope of this work, and results from the existence of a *bing bang* bifurcation. This is discussed in the next section for first and second order systems.

## Big Bang Bifurcation of the Period Adding Type

### The 1D Case

Let us first study the 1D case ($n = 1$ and $G_c(s) = 1$) when (3) is a scalar equation, which was reported in [4]. After applying the change of variable $z = y - y_c$ to the original system, the sliding surface is given by $z = 0$, and the map (7) becomes

$$\tilde{P}(z) = \begin{cases} \tilde{P}_\ell(z) := \rho z + \mu_\ell & \text{if } z < 0 \\ \tilde{P}_r(z) := \rho z + \mu_r & \text{if } z > 0 \end{cases} \tag{9}$$

with

$$\rho = e^{a_0 T} < 1, \quad \mu_r = (\rho - 1)\left(y_c - \frac{bk}{a_0}\right) \in \mathbb{R} \text{ and } \mu_\ell = (\rho - 1)\left(y_c + \frac{bk}{a_0}\right) \in \mathbb{R}, \tag{10}$$

and the fixed points

$$z_r^* = -\frac{\mu_r}{\rho - 1} \in \mathbb{R}, \quad z_\ell^* = -\frac{\mu_\ell}{\rho - 1} \in \mathbb{R}. \tag{11}$$

In order to describe the dynamics of the map (9) when both fixed points of its branches are virtual, $\mu_r < 0$ and $\mu_\ell > 0$, we first focus on the bifurcations that occur in their transition from virtual to feasible or vice versa. To this end, we first restrict ourselves to a suitable 2D parameter space, in terms of (11), where the position of $z_r^*$ and $z_\ell^*$ with respect to the boundary $z = 0$ can be independently represented. More precisely, we are interested on the existence of two curves such that a variation of the parameters along them, affects only the position of one fixed point. We proceed arguing with the parameters $\rho$, $\mu_r$ and $\mu_\ell$ in (9), although we will later translate our discussion to the original parameters $a$, $T$, $b$, $k$ and $y_c$.

We remark that we benefit from the linearity of the system in order to perform explicit calculations, although the same argumentations below hold also for a non-linear system.

As these transitions occur when one of the fixed points collides with the boundary $z = 0$, these are given by border collision bifurcations. Although the parameter $\rho$ influences on the position of both fixed points, as we are restricted to $0 < \rho < 1$, its variation does not lead to such type of bifurcations. Hence, we focus on the $\mu_\ell \times \mu_r$ parameter space.

There, the vertical and horizontal axis represent border collision bifurcation curves that the fixed points, $\mu_\ell^*$ and $\mu_r^*$, undergo. Of particular interest is the origin of this parameter space, which is a co-dimension two bifurcation point, as both fixed points simultaneously collide with the boundary. Depending on the sign of the eigenvalues associated with the colliding fixed points, such a point may become a *big bang* bifurcation point, where an infinite number of (border collision) bifurcation curves emanate from. If this occurs, these bifurcation curves separate existence regions of periodic orbits located at the region in the parameter space $\mu_\ell \times \mu_r$ where both fixed points are virtual.

This basically depends on the sign of the eigenvalues of the colliding fixed points, which, for the 1D case, are the slopes of the map near the discontinuity. The possible bifurcation scenarios for a 1D contracting linear map with one discontinuity were described in [2] through numerical observations, and were generalized in [1] (see also the bibliography reported there). With independence of the particular nature of the map, it was proven there that when the sign of the eigenvalues associated with the colliding fixed points are different (also known as increasing-decreasing/decreasing-increasing case), then a big bang bifurcation of the *period incrementing* type occurs. It was also suggested that, when both are positive (increasing-increasing case), a *period adding* big bang bifurcation (described below) occurs. Although this result was conjectured, the resulting bifurcation scenario has been highly reported in the literature [14,3,6,7,9,8,12,16,15], and hence it is a well accepted result.

In our case, as the eigenvalues associated with the colliding fixed points are positive, $0 < \rho < 1$, a big bang of the period adding type occurs at the origin of the parameter space $\mu_\ell \times \mu_r$.

This implies that the region located near the origin of the parameter space where both fixed points are virtual is fully covered by an infinite number of regions where a unique periodic orbit exists. All these regions collapse at the origin and, hence, all the possible periodic orbits exist for any arbitrarily small neighbourhood containing the origin of the parameter space.

To understand how these periodic orbits are organized, let us introduce the following symbolic codification (see [2] for a more extended explanation). Let $(z_1, \ldots, z_n)$ be the sequence of points forming a periodic orbit of period $n$, then we consider the symbolic sequence obtained by replacing each of these points by $\mathcal{L}$ if $z_i < 0$ and $\mathcal{R}$ if $z_i > 0$. Then, the symbolic sequences of the periodic orbits are obtained by a gluing process between periodic orbits. More precisely, in-between the regions of existence of two periodic orbits of periods $n$ and $m$ with symbolic sequences $\alpha$ and $\beta$ one finds a region where the $(n+m)$-periodic orbit with symbolic sequence $\alpha\beta$ (their concatenation) exists. As the symbolic sequences are glued, the periods are added, and hence this scenario was referred in [2] as period adding. This process starts with the fixed points $z_\ell^* \to \mathcal{L}$ and $z_r^* \to \mathcal{R}$, which are "glued" to form the 2-periodic orbit $\mathcal{LR}$, and is repeated add infinitum. Thus, in any arbitrarily small neighbourhood of the origin of the parameter space one can find an infinite number of periodic orbits with arbitrarily large periods.

Let us adapt the situation described before in terms of he parameters involved in the original system (3–4) for $n = 1$. Let us first focus on their influence on the dynamics of the map (9).

As it comes from the relations shown in (10), the most relevant parameters regarding the influence on the location of the fixed points $z_\ell^*$ and $z_r^*$ are $y_c$, $k$ and $b$. We proceed arguing with the pair $(y_c, k)$, as they are the parameters to be tunned and, hence, are of more interest from the control design point of view. However, the following discussion can be easily extended to the pair $(y_c, b)$.

In this parameter space, the lines

$$k = -a_0/by_c \text{ and } k = a_0/by_c \tag{12}$$

represent border collision bifurcation curves for $z_\ell^*$ and $z_r^*$, respectively. Hence, as both are attracting with positive associated eigenvalues $(0 < \rho < 1)$, a big bang bifurcation of the period adding type occurs at the intersection of these lines, $(y_c, k) = (0, 0)$, where two border collision bifurcation simultaneously occur.

Note that, although the parameter $a_0$ also influences on the position of the fixed points, it comes that a big bang bifurcation may occur for $a_0 = 0$. However, for such a value the fixed points are no longer attractive and, hence, the results obtained so far on big bang bifurcation can not be applied.

To demonstrate this, we show in Fig. 2a the bifurcation scenario in the $y_c \times k$ parameter space, where one can observe the infinite number of bifurcation curves emanating from the origin. The adding scenario is presented in Fig. 2b, where the periods of the periodic orbits found along the curve marked in Fig. 2a are shown.

It comes from the adding procedure described above that all the periodic orbits step at both sides of the boundary $z = 0$. Hence, each of these $n$-periodic orbits correspond in the original 1D continuous model (3–4) to a continuous $nT$-periodic orbit which oscillates around $y = y_c$. In addition, the amplitude of all these orbits tend to zero as the parameters $y_c$ and $k$ get close to the big bang bifurcation.

## A Second Order System

We now extend the results shown in the previous section to a second order system.

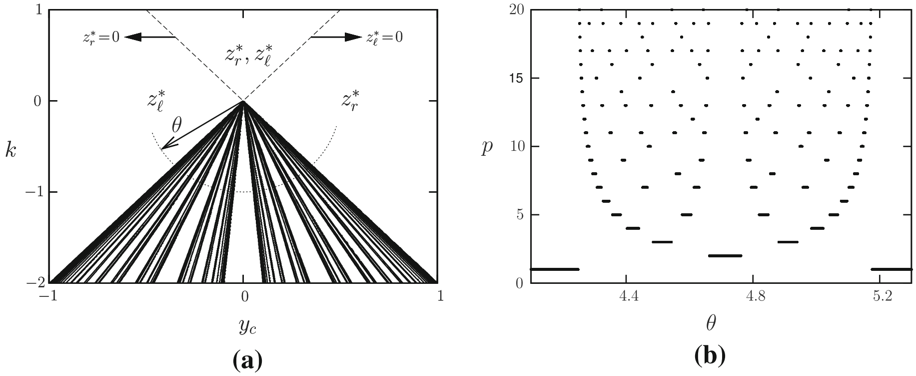


**Fig. 2** **a** Big bang bifurcation in the $(y_c, k)$ parameter space for $a_0 = -2, b = 1$ and $T = 0.1$. The fixed points $z_i^*$ are labeled in the regions where they are feasible, and, as *dashed lines*, the border collision bifurcation curves where they become virtual are shown. In **b** we show the periods $p$ of the periodic orbits found along the pointed curve in (**a**), which is parametrized by the angle $\theta$

In this case, we have

$$A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \ \bar{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \ \sigma = y_1 - y_c + c_1 y_2,$$

where $a_0$ and $a_1$ are such that the eigenvalues of the matrix $A$ have real negative eigenvalues.

For commodity, in order to easily proceed as before and argue with the relative position of the fixed points $\bar{z}_\ell^*$ and $\bar{z}_r^*$ with respect the boundary, we introduce new coordinates $\bar{z} = (z_1, z_2)$ given by

$$\bar{z} = \underbrace{\begin{pmatrix} 1 & c_1 \\ a_0 c_1 - a_1 & 1 \end{pmatrix}}_{\phi} - \begin{pmatrix} y_c \\ 0 \end{pmatrix}.$$

Note that one can always perform such a change of variables as long as the vector $(1, c_1)^T$ is not an eigenvector of the matrix $A$.

In these new variables, the map (7) becomes

$$\tilde{P}(\bar{z}) = \begin{cases} \tilde{P}_\ell(\bar{z}) := \tilde{\rho}\bar{z} + \bar{\mu}_\ell & \text{if } z_1 < 0 \\ \tilde{P}_r(\bar{z}) := \tilde{\rho}\bar{z} + \bar{\mu}_r & z_1 > 0, \end{cases} \tag{13}$$

where

$$\tilde{\rho} = e^{\tilde{A}T}, \ \tilde{A} = \begin{pmatrix} -a_1 & -1 \\ a_0 & 0 \end{pmatrix}$$

$$\bar{\mu}_\ell = (\tilde{\rho} - Id)\tilde{A}^{-1}\left(-\phi\begin{pmatrix} 0 \\ kb \end{pmatrix} + \tilde{A}\begin{pmatrix} y_c \\ 0 \end{pmatrix}\right)$$

$$\bar{\mu}_r = (\tilde{\rho} - Id)\tilde{A}^{-1}\left(\phi\begin{pmatrix} 0 \\ kb \end{pmatrix} + \tilde{A}\begin{pmatrix} y_c \\ 0 \end{pmatrix}\right).$$

The main advantage of this change of variables consists on the fact that the boundary becomes $z_1 = 0$, independently of the parameters, while the matrix $\tilde{\rho}$ remains only dependent on the parameters $a_i$. Hence, the relevant parameters for the study of the border collision bifurcations only influence the position of the fixed points, which become

$$\bar{z}_\ell^* = \begin{pmatrix} -y_c - \frac{kb}{a_0} \\ -\frac{kb}{a_0}(a_0c_1 - a_1) \end{pmatrix}$$

$$\bar{z}_r^* = \begin{pmatrix} -y_c + \frac{kb}{a_0} \\ \frac{kb}{a_0}(a_0c_1 - a_1) \end{pmatrix}$$

The border collision bifurcation curves that the fixed points undergo become the same expressions as in the 1D case, given in (12). Hence, arguing again in the $y_c \times k$ parameter space, for $y_c = k = 0$ both fixed points simultaneously collide with the boundary $z_1 = 0$ and become virtual.

We now conjecture an extension to 2D maps of the result used above for the 1D maps. In the considered situation regarding the simultaneously collision of attracting fixed points with the boundary, there exist a big bang bifurcation of the period adding type if there exist an open neighbourhood $\mathcal{U}$ such that, at the simultaneous collision,

$$\bar{z}_\ell^*, \bar{z}_r^* \in \mathcal{U}$$

$$\tilde{P}_i(\mathcal{U} \cap \mathcal{X}_i) \subset \mathcal{U} \cap \mathcal{X}_i, \; i \in \{\ell, r\},$$

where $\mathcal{X}_\ell$ and $\mathcal{X}_r$ are the left and right part of $\mathbb{R}^2$ separated by the boundary.

In our case, these conditions coincide with the sliding conditions given in (5). This is because, at the big bang bifurcation point, $k = y_c = 0$, both fixed points collide with origin of the state space, $\bar{z}_i^*(0, 0)^T$. Hence, near the bifurcation point, $|u_{eq}| << 1$ and thus condition (6) is fulfilled if

$$c_1 \neq 0.$$

Note that, although the map (13) is continuous at $\bar{z} = (0, 0)^T$ at the big bang bifurcation point because both fixed points coincide at $\bar{z} = (0, 0)^T$, continuity is not assumed in the conditions mentioned above.

The simulations shown in Fig. 3 show how a big bang bifurcation of the period adding type occurs for $y_c = k = 0$ if $c_1 \neq 0$.

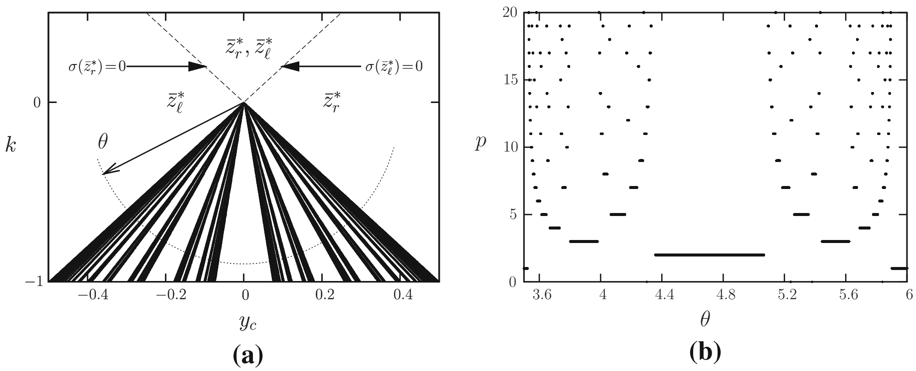


**Fig. 3** Big bang bifurcation in the $(y_c, k)$ parameter space for $a_0 = -2$, $a_1 = -5$, $b = 1$, $c_1 = 1.5$ and $T = 0.1$. In **a** we show the border collision bifurcation curves separating existence regions of periodic orbits. In **b** the periods of the periodic orbits found along the pointed curve in (**a**) parametrized by the angle $\theta$

## Conclusions

A co-dimension-2 big bang bifurcation in a 2D parameter space relevant from the control design point of view has been presented in a class of first and second order systems with a relay. While its occurrence in the 1D case is based on results given in the literature, sufficient conditions have been suggested for an extension to the 2D case.

**Acknowledgements** This work has been partially supported by DAAD grant A/08/11804, MICINN-FEDER grants MTM2009-06973, DPI2008-01408 and DPI2010-15110.

## References

1. Avrutin, V., Granados, A., Schanz, M.: Sufficient conditions for a period increment big bang bifurcation in one-dimensional maps. Nonlinearity **24**(9), 2575–2598 (2011)
2. Avrutin, V., Schanz, M.: On multi-parametric bifurcations in a scalar piecewise-linear map. Nonlinearity **19**, 531–552 (2006)
3. Coullet, P.C., Gambaudo, J.M., Tresser, C.: Une nouvelle bifurcation de codimension 2: le colage de cycles. C. R. Acad. Sc. Paris, série I **299**, 253–256 (1984)
4. Fossas, E., Granados, A.: Big bang bifurcations in a first order systems with a relay. In: Olejnik, P., Mrozowski, J., Awrejcewicz, J., Kaźmierczak, M. (eds.) Proceedings of 11th conference of dynamical systems. Analytical/numerical methods, stability, bifurcation and chaos, Lodz, pp. 147–152 (2011)
5. Galias, Z.: Basins of attraction for periodic solutions of discretized sliding mode control systems. In: Proceedings of 2010 IEEE International Symposium on Circuits and Systems (ISCAS), New Port Beach (2010)
6. Gambaudo, J.M., Glendinning, P., Tresser, C.: Collage de cycles et suites de Farey. C. R. Acad. Sc. Paris, série I **299**, 711–714 (1984)
7. Gambaudo, J.M., Glendinning, P., Tresser, T.: The gluing bifurcation: symbolic dynamics of the closed curves. Nonlinearity **1**, 203–214 (1988)
8. Ghrist, R., Holmes, P.J. : Knotting within the gluing bifurcation. In: Thompson, J., Bishop, S. (ed.) IUTAM symposium on nonlinearilty and chaos in the enginnering dynamics., pp. 299–315. John Wiley Press, Chichester (1994)
9. Gambaudo, J.M., Procaccia, I., Thomae, S., Tresser, C.: New universal scenarios for the onset of chaos in Lorenz-type flows. Phys. Rev. Lett **57**, 925–928 (1986)
10. Galias, Z., Yu, X.: Analysis of zero-order holder discretization of two-dimensional sliding mode control systems. IEEE Trans. Circuits Syst. II **55**(12), 1269–1273 (2008)
11. Galias, Z., Yu, X.: Study of periodic solutions of discretized two-dimensional sliding mode control systems. IEEE Trans. Circuits Syst. II **58**(6), 381–385 (2011)
12. Homburg, A.J.: Global aspects of homoclinic bifurcations of vector fields. Mem. Am. Math. Soc. **121**, 578 (1996)
13. Kabe, T., Saito, T., Torikai, H.: Analysis of piecewise constant models of power converters. In: International symposium on nonlinear theory and its applications, NOLTA, Dresden, pp. 71–74, (2004)
14. Leonov, N.N.: On a pointwise mapping of a line into itself. Radiofisika, **2**(6), 942–956 (1959) (in Russian)
15. Procaccia, I., Thomae, S., Tresser, C.: First-return maps as a unified renormalization scheme for dynamical systems. Phys. Rev. A **35**, 1884–1900 (1987)
16. Turaev, D.V., Shil'nikov, L.P.: On bifurcations of a homoclinic "Figure of Eight" for a saddle with a negative saddle value. Sov. Math. Dokl., **34**, 397–401 (1987) (Russian version 1986)
17. Utkin Vadim, I.: Variable structure systems with sliding modes. IEEE Trans. Autom. Control **ac-22**(2), 212–222 (1977)
18. Wang, B., Yu, X., Li, X.: ZOH discretization effect on high-order sliding mode control systems. IEEE Trans. Ind. Electron. **55**(11), 4055–4064 (2008)
19. Yu, X., Chen, G.: Discretization behaviors of equivalent control based sliding mode control systems. IEEE Trans. Autom. Control **48**(9), 1641–1646 (2003)