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Big bang bifurcations in a first order system with a relay

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In this work we describe the bifurcation scenario found in a general first order system when a relay based proportional control is considered. We show the existence of a big bang bifurcation causing the existence of an infinite number of period orbits near a co-dimension two bifurcation point.

1. Introduction

There exist many methods in order to force a system to exhibit a certain desired behaviour. If its output is required to be near a certain value, one common strategy consists on implementing a control system such that two different actions are applied depending on the sign of a certain switching function which depends on the actual state. In the one-dimensional case, this may be the difference between the actual output and the desired one. In general, this leads to a non-smooth system which, among other phenomena, in larger dimensions, can exhibit sliding.

As, in practice, the states are sampled at particular values of time, one considers a discretization of such a construction through a zero order holder keeping the sampled value constant until the next sampling. Instead of a differential equation, the system is then usually modeled by a map, whose dynamics may differ completely from the time continuous system assuming a "continuous sampling" of the states (absence of the zero order holder). This occurs especially when the states are close to the switching values, as this map does not coincide with the stroboscopic Poincaré map of the time-continuous system. In particular, new bifurcation phenomena may be introduced. This is the case of the so-called *big bang* bifurcations, first introduced in 1. when simulating a piecewise-linear system and later more understood and generalized in 2. These were also detected in numerical studies when analyzing power converters in 5. In this work, we use the results in 2. to show and predict their existence in a first order control system as the one described above. We also perform numerical simulations for such a system reproducing the bifurcation scenarios shown in 1.

2. A first order system with a relay based control

2.1. System description

Let us consider the control system shown in Fig. 1.

In order to control the first order plant $G_s(s) = b/(s-a)$ and make its output be close to

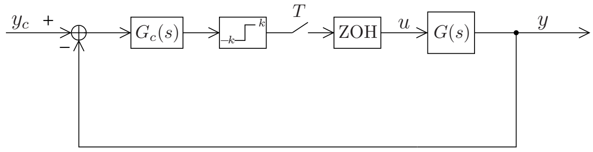


Figure 1. A linear system control by a relay.

a certain desired value y_c, we consider a control system based on a relay of gain k with a proportional controller $G_c(s) = k_p$. That is, the sign of the signal $k_p(y_c - y(t))$ is evaluated every a certain time T, and the input, u, of the plant is set to k or $-k$ if it is positive or negative, respectively. This is then kept constant until the next sampling instant. Note that the parameter k_p does not play any role except for its sign, which, without loss of generality, we will assume positive.

Hence, the dynamical system that models the control system can be written as

$$\dot{y} = ay + bu \tag{1}$$

with

$$u = \begin{cases} k & \text{if } y(nT) > y_c \\ -k & \text{if } y(nT) < y_c \end{cases} \qquad t \in [nT, (n+1)T), \tag{2}$$

which is a piecewise-smooth system with $y = y_c$ as boundary or switching manifold.

In order to make the system stable and permit it to be close to the desired value y_c, we assume $a < 0$ and

$$-\frac{bk}{a} < y_c < \frac{bk}{a}. \tag{3}$$

For convenience, we perform the change of variables $z = y - y_c$ and set the switching value at $z = 0$. Thus, the dynamical system becomes

$$\dot{z} = az + w \tag{4}$$

with

$$w = \begin{cases} bk + ay_c & \text{if } z > 0 \\ -bk + ay_c & \text{if } z < 0, \;\; t \in [nT, (n+1)T]. \end{cases} \tag{5}$$

Hence the goal is now to stabilize system (4)-(5) around $z = 0$, which becomes possible due to conditions (3) and the fact that $a < 0$.
As the sampling performed by the switching process is T-periodic, we proceed as usual and consider discretization of the system. To this end, we consider the Poincaré map at return time T associated to the flow obtained by fixing w to $bk + ay_c$ or to $-bk + ay_c$ depending on whether the initial condition is positive or negative, respectively. Hence, using that the solution of a linear system $\dot{z} = \widehat{a}z + \widehat{b}$ such that $z(0) = z_0$ is $z(t) = e^{\widehat{a}t}z_0 + (e^{\widehat{a}t} - 1)\frac{\widehat{b}}{\widehat{a}}$, we obtain the discrete dynamical system $z_{n+1} = P(z_n)$ where

$$P(z) = \begin{cases} P_r(z) := \rho z + \mu_r & \text{if } z > 0 \\ P_\ell(z) := \rho z + \mu_\ell & \text{if } z < 0 \end{cases} \tag{6}$$

with

$$\rho = e^{aT}, \quad \mu_r = (\rho - 1)(y_c - \frac{bk}{a}) \text{ and } \mu_\ell = (\rho - 1)(y_c + \frac{bk}{a}). \tag{7}$$

2.2. System dynamics

Each branch of the map (6), P_r and P_ℓ, has a fixed point

$$z_r^* = -\frac{\mu_r}{\rho - 1}, \quad z_\ell^* = -\frac{\mu_\ell}{\rho - 1}, \tag{8}$$

which may be *feasible* or *virtual* depending on whether they belong to the domains of their respective maps or not, respectively.

Regarding the possible dynamics, we distinguish between three situations.
If both fixed points are feasible ($z_r^* > 0$ and $z_\ell^* < 0$) they also become fixed points of the map (6). Hence, as $0 < \rho < 1$ ($a < 0$), both are attracting fixed points and all the trajectories tend towards one of them, with domains of attraction $z > 0$ and $z < 0$ for z_r^* and z_ℓ^*, respectively.
If only one of both fixed points is feasible ($z_r^* < 0$ and $z_\ell^* < 0$ or vice versa), then it becomes the unique fixed point of the map (6). As $0 < \rho < 1$, all trajectories tend towards it, and its domain of attraction becomes $\mathbb{R}$.
Note that, in these two previous cases, the control specification is not fulfilled as the asymptotic dynamics is away from $z = 0$.

The third situation occurs when both fixed points are virtual ($z_r^* < 0 \quad z_\ell^* > 0$) which precisely coincides with the conditions shown in Eq. (3). In this case, the possible dynamics consists on periodic orbits which may possess arbitrarily large periods and whose iterates step on both sides of the boundary $z = 0$. Properly tunning the parameters, the amplitude of all these orbits can be chose arbitrarily small. Thus, in this case, the design conditions of the control are satisfied, as the asymptotic dynamics are close to $z = 0$ (y close to y_c). A precise description of all the possible periodic orbits is precisely the scope of this work, and results from the existence of a *bing bang* bifurcation. This is discussed in the next section.

3. Big bang bifurcation of the period adding type

3.1. General description

In order to describe the dynamics of the map (6) when both fixed points of its branches are virtual, we first focus on the bifurcations that occur in their transition from virtual to feasible or vice versa. To this end, we first restrict ourselves to a suitable two-dimensional parameter space, in terms of (8), where the position of z_r^* and z_ℓ^* with respect to the boundary $z = 0$ can be independently represented. More precisely, we assume the existence of two curves such that a variation of the parameters along them, affects only the position of one fixed point. We proceed arguing with the parameters ρ, μ_r and μ_ℓ in (6), although we will later translate our discussion to the original parameters a, T, b, k and y_c.
We remark that we benefit from the linearity of the system in order to perform explicit calculations, although the same argumentations below hold also for a non-linear system.

As these transitions occur when one of the fixed points collides with the boundary $z = 0$, these are given by border collision bifurcations. Although the parameter ρ influences on the position of both fixed points, as we are restricted to $0 < \rho < 1$, its variation does not lead to such type of bifurcations. Hence, we focus on the $\mu_\ell \times \mu_r$ parameter space.
There, the vertical and horizontal axis represent border collision bifurcation curves that the fixed points, μ_ℓ^* and μ_r^*, undergo. Of particular interest is the origin of this parameter space, which is a co-dimension two bifurcation point, as both fixed points simultaneously collide with the boundary. Depending on the sign of the eigenvalues associated with the colliding fixed points, such a point may become a *big bang* bifurcation point, where an infinite number of (border collision) bifurcation curves emanate from. If this occurs, these bifurcation curves separate existence regions of periodic orbits located at the region in the parameter space $\mu_\ell \times \mu_r$ where both fixed points are virtual.
In this case, as the eigenvalues associated with the colliding fixed points are positive, $0 < \rho < 1$, then a big bang of the period adding type occurs (see 2.). This implies that the region in the parameter space where both fixed points are virtual is fully covered by an infinite number of regions where a unique periodic orbit exists. All these regions collapse at the origin and, hence, all the possible periodic orbits exist for any arbitrarily small neighbourhood containing the origin of the parameter space.

To understand how these periodic orbits are organized, let us introduce the following symbolic codification (see 1. for a more extended explanation). Let $(z_1, \dots, z_n)$ be the sequence of points forming a periodic orbit of period n, then we consider the symbolic sequence obtained by replacing each of these points by $\mathcal{L}$ if $z_i < 0$ and $\mathcal{R}$ if $z_i > 0$. Then, the symbolic sequences of the periodic orbits are obtained by a gluing process between periodic orbits. More precisely, in-between the regions of existence of two periodic orbits of periods n and m with symbolic sequences σ and γ one finds a region where the $(n+m)$-periodic orbit with symbolic sequence $\sigma\gamma$ (their concatenation) exists. As the symbolic sequences are glued, the periods are added, and hence this scenario is referred in 1. as period adding. This process starts with the fixed points $z_\ell^* \to \mathcal{L}$ and $z_r^* \to \mathcal{R}$, which are "glued" to form the 2-periodic orbit $\mathcal{LR}$, and is repeated add infinitum. Thus, in any arbitrarily small neighbourhood of the origin of the parameter space one can find an infinite number of periodic orbits with arbitrarily large periods.

3.2. Interpretation in the original parameters and simulations

Let us adapt the situation described before into the parameters of the original system (1)-(2). Let us first focus on their influence on the dynamics of the map (6).

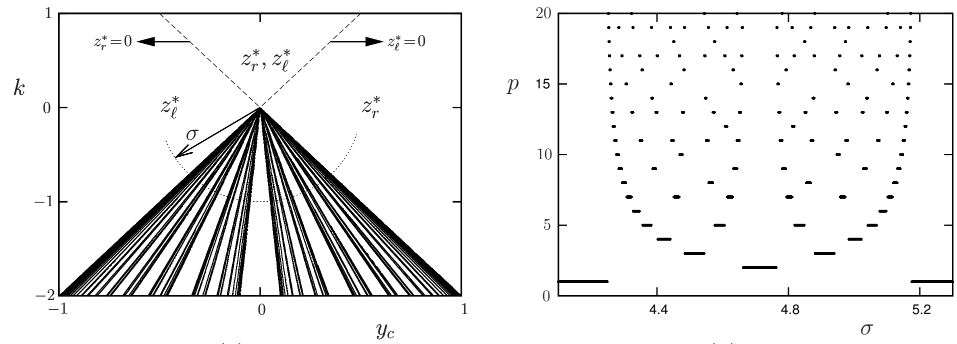
As it comes from the relations shown in (7), the most relevant parameters regarding the influence on the location of the fixed points z_ℓ^* and z_r^* are y_c, k and b. We proceed arguing with the pair (y_c, k), as they are the parameters to be tunned and, hence, are of more interest from the control design point of view. However, the following discussion can be easily extended to any pair in (y_c, k, b).

In this parameter space, the lines $k = -a/by_c$ and $k = a/by_c$ represent border collision bifurcation curves for z_ℓ^* and z_r^*, respectively. Hence, as both are attracting with positive associated eigenvalues $(0 < \rho < 1)$, a big bang bifurcation of the period adding type occurs at the intersection of these lines, $(y_c, k) = (0, 0)$, where two border collision bifurcation simultaneously occur.

Note that, although the parameter a also influences on the position of the fixed points, it comes that a big bang bifurcation may occur for $a = 0$. However, for such a value the fixed points are no longer attractive and, hence, the results on big bang bifurcation can not be applied.

To demonstrate this, we show in Fig. 2(a) the bifurcation scenario in the $y_c \times k$ parameter space, where one can observe the infinite number of bifurcation curves emanating from the origin. The adding scenario is presented in Fig. 2(b), where the periods of the periodic orbits found along the curve marked in Fig. 2(a) are shown.

It comes from the adding procedure described above that all the periodic orbits step at both



(a) (b)

Figure 2. (a) Big bang bifurcation in the (y_c, k) parameter space for $a = -2$, $b = 1$ and $T = 0.1$. The fixed points z_i^* are labeled in the regions where they are feasible, and, as dashed lines, the border collision bifurcation curves where they become virtual are shown. The periods of the periodic orbits found along the pointed curve parametrized by σ are shown in (b).

sides of the boundary $z = 0$. Hence, each of these n-periodic orbits correspond in the original continuous model (1)-(2) to a continuous nT-periodic orbit which oscillates around $y = y_c$. In addition, the amplitude of all these orbits tend to zero as the parameters y_c and k get close to the big bang bifurcation.

4. Conclusions

A co-dimension-2 big bang bifurcation has been presented in a first order system with a relay in the two-dimensional parameter space relevant from the control design point of view.

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