SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 11, No. 3, pp. 801–830

The Melnikov Method and Subharmonic Orbits in a Piecewise-Smooth System*

A. Granados†, S. J. Hogan‡, and T. M. Seara§

Abstract. We consider a two-dimensional piecewise-smooth system defined in two domains separated by a switching manifold Σ. We assume that there exists a piecewise-defined continuous Hamiltonian that is a first integral of the system. We also suppose that the system possesses an invisible fold-fold at the origin and two heteroclinic orbits connecting two hyperbolic critical points on either side of Σ. Finally, we assume that the region enclosed by these heteroclinic connections is fully covered by periodic orbits surrounding the origin, whose periods monotonically increase as they approach the heteroclinic connection. For a nonautonomous (T-periodic) Hamiltonian perturbation of amplitude ε, we rigorously prove, for every n and m relatively prime and $\varepsilon > 0$ small enough, that there exists an nT-periodic orbit impacting $2m$ times with the switching manifold at every period if a modified subharmonic Melnikov function possesses a simple zero. We also prove that if the orbits are discontinuous when they cross Σ, then all these orbits exist if the relative size of $\varepsilon > 0$ with respect to the magnitude of this jump is large enough. In addition, we obtain similar conditions for the splitting of the heteroclinic connections.

Key words. subharmonic orbits, heteroclinic connections, piecewise-smooth impact systems, Melnikov method

AMS subject classifications. 37, 37J45, 37N15

DOI. 10.1137/110850359

1. Introduction.

The Melnikov method provides tools for determining the persistence of periodic orbits and homoclinic/heteroclinic connections for planar regular systems under non-autonomous periodic perturbations [GH83]. This persistence is guaranteed by the existence of simple zeros of the subharmonic Melnikov function and the Melnikov function, respectively. In this work we extend these classical results to a class of piecewise-smooth differential equations, which generalize a mechanical impact model. In such systems, the perturbation typically models an external forcing and, hence, affects a second order differential equation. In this paper, we allow for a general periodic Hamiltonian perturbation, potentially influencing both velocity and acceleration. Note that no symmetry is assumed in either the perturbed or unperturbed system.

The unperturbed system is defined in two domains separated by a switching manifold Σ and possesses one hyperbolic critical point on either side of Σ. We distinguish between two different unperturbed systems. In the first case, which we call conservative, two heteroclinic

*Received by the editors October 5, 2011; accepted for publication (in revised form) by P. Holmes April 6, 2012; published electronically July 10, 2012. This work was partially supported by DAAD grant A/08/11804, MICINN-FEDER grant MTM2009-06973, and EPSRC grant EP/E032249/1.
http://www.siam.org/journals/siads/11-3/85035.html
†IPVS, Universität Stuttgart, Stuttgart 70569, Germany (albert.granados@ipvs.uni-stuttgart.de).
‡Department of Engineering Mathematics, University of Bristol, Bristol BS8 1TR, United Kingdom (s.j.hogan@bristol.ac.uk).
§Departament de Matematica Aplicada 1, Universitat Politècnica de Catalunya, Barcelona 08028, Spain (tere.m-seara@upc.edu).

trajectories connect both hyperbolic points and surround a region completely covered by periodic orbits including the origin. In the second case, we introduce an energy dissipation, which is modeled by an algebraic condition that forces the solutions to undergo a discontinuity every time they cross the switching manifold. Then the origin becomes a global attractor, and nontrivial periodic orbits and homoclinic/heteroclinic connections cannot exist for the unperturbed system.

In order to consider the persistence of periodic orbits for a smooth system, the classical Melnikov method looks for fixed (or periodic) points of the time T stroboscopic map, where T is the period of the perturbation. Since this map is as regular as the flow, one can study its periodic points using classical perturbation methods.

However, for our class of systems, the time T stroboscopic map becomes unwieldy to use because one has to check the number of times that the flow crosses the switching manifold, which is a priori unknown and can even be arbitrarily large. Hence, the regularity properties of this map are not straightforward. Instead of the classical stroboscopic map, using the switching manifold as a Poincaré section and adding time as variable, we consider the first return Poincaré map, the so-called *impact map*. For the system under consideration, the unperturbed impact map is defined on the cylinder and, under generic hypothesis, is a twist map. Moreover, this map is smooth, and hence we can use classical perturbation theory to rigorously prove sufficient conditions for the existence of periodic orbits by looking for periodic points of the perturbed impact map. In the conservative case, these conditions turn out to be the same ones given by the classical Melnikov method, thus extending it to a class of piecewise-smooth systems (Theorem 3.1). In addition, we rigorously prove that the simple zeros of the subharmonic adapted Melnikov function can guarantee the existence of periodic orbits when the trajectories are discontinuous (Theorem 3.3).

The impact map can also be used to prove the existence of invariant KAM tori in the system since it is a twist map in the unperturbed case. After writing the system in action-angle variables, these ideas were applied in [KKY97] to a different system to prove the existence of such tori.

The use of perturbation methods for the existence of periodic orbits of some specific linear systems can be found in [TA07, CFGF11]. Similar methods have also been applied in [DLZ08, LH10, DL12] to general autonomous systems for the persistence of periodic orbits. The existence of subharmonic orbits in a class of piecewise-smooth systems is proved in [Yag] by uniformly approximating the solutions of the piecewise-smooth system with solutions of a smooth one.

The proof of the persistence of heteroclinic/homoclinic connections for periodically perturbed smooth systems is well established by the classical Melnikov method [GH83]. The main idea is to take some point on the unperturbed homoclinic/heteroclinic connection and consider a section normal to the unperturbed vector field at this point. By the regularity properties of the stable and unstable manifolds of hyperbolic critical points in smooth systems, one can measure the distance between the perturbed manifolds. The Melnikov method derives a first order asymptotic formula for this distance given by the so-called Melnikov function, which is a periodic function whose zeros give, up to first order in the perturbation parameter, the perturbed homoclinic/heteroclinic points.

By contrast, since the vector normal to the unperturbed vector field is not defined everywhere in the piecewise-defined system considered here, we proceed as in [BK91, Hog92] and look for the intersection between the stable and unstable manifolds with the switching manifold. Since this intersection depends smoothly on the perturbation parameter, we obtain an asymptotic formula for the distance between the manifolds in this section, which turns out to be a Melnikov function modified for the piecewise-smooth case. The zeros of this Melnikov function give rise to the existence of heteroclinic connections for the perturbed system. Therefore, we rigorously extend the classical Melnikov method for heteroclinic connections to this class of piecewise-smooth systems.

When the loss of energy is considered the zeros of the Melnikov function can be used to guarantee the existence of transversal heteroclinic intersections. Both results are given in Theorem 4.1.

Other work [Kun00, Kuk07, BF08, BF11] has considered the extension of the Melnikov method to piecewise-defined systems. In these papers, the stable and unstable manifolds of a hyperbolic point located on one side of the switching manifold intersect it at two points that are connected by a trajectory defined on the other side of the switching manifold, thus forming a homoclinic loop for the unperturbed system. Then persistence is related with the zeros of a modified Melnikov function by proving the existence of solutions of a boundary value problem. A Melnikov method for some classes of nonlinear impact oscillators is developed in [DZ05, XFR09].

This paper is organized as follows. In section 2, we describe the class of system that we consider and introduce some notation and tools needed for this work. In section 3, we prove the existence of periodic orbits distinguishing between the conservative and dissipative cases. Section 4 is devoted to heteroclinic connections. Finally, in section 5, we use the example of the rocking block to illustrate the results obtained regarding the periodic orbits and compare this with the work of [Hog89].

2. System description.

2.1. General system definition. We divide the plane into two sets (see Figures 1–2),

$$S^+ = \{(x,y) \in \mathbb{R}^2 \,|\, x > 0\},$$
$$S^- = \{(x,y) \in \mathbb{R}^2 \,|\, x < 0\},$$

separated by the switching manifold

$$\Sigma = \Sigma^+ \cup \Sigma^- \cup (0,0), \tag{2.1}$$

where

$$\Sigma^+ = \Big\{(x,y) \in \mathbb{R}^2 \,|\, x = 0,\ y > 0\Big\},$$
$$\Sigma^- = \Big\{(x,y) \in \mathbb{R}^2 \,|\, x = 0,\ y < 0\Big\}.$$

We consider the piecewise-smooth system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \mathcal{X}_0^+(x,y) + \varepsilon \mathcal{X}_1^+(x,y,t) & \text{if } (x,y) \in S^+, \\ \mathcal{X}_0^-(x,y) + \varepsilon \mathcal{X}_1^-(x,y,t) & \text{if } (x,y) \in S^-. \end{cases} \tag{2.2}$$

We assume $\mathcal{X}_0^{\pm} \in C^{\infty}(\mathbb{R}^2)$ and $\mathcal{X}_1^{\pm}(x, y, t) \in C^{\infty}(\mathbb{R}^3)$, although this can be relaxed to less regularity in $S^{\pm}$ and $S^{\pm} \times \mathbb{R}$, respectively.

System (2.2) is a Hamiltonian system associated with a C^0 piecewise-smooth Hamiltonian of the form

$$H_{\varepsilon}(x, y, t) = H_0(x, y) + \varepsilon H_1(x, y, t). \tag{2.3}$$

The unperturbed $C^0(\mathbb{R}^2)$ Hamiltonian H_0 is a classical Hamiltonian given by

$$H_0(x, y) := \frac{y^2}{2} + V(x) := \begin{cases} H_0^+(x, y) := \dfrac{y^2}{2} + V^+(x) & \text{if } (x, y) \in S^+ \cup \Sigma, \\ H_0^-(x, y) := \dfrac{y^2}{2} + V^-(x) & \text{if } (x, y) \in S^-, \end{cases} \tag{2.4}$$

with $V^{\pm} \in C^{\infty}(\mathbb{R})$ satisfying $V^+(0) = V^-(0)$.

Similarly, the nonautonomous T-periodic $C^0(\mathbb{R}^3)$ perturbation, εH_1, is given by

$$H_1(x, y, t) := \begin{cases} H_1^+(x, y, t) & \text{if } (x, y) \in S^+ \cup \Sigma^+, \\ H_1^-(x, y, t) & \text{if } (x, y) \in S^- \cup \Sigma^- \end{cases}$$

satisfying $H_1^+(0, y, t) = H_1^-(0, y, t)$ for all $(y, t) \in \mathbb{R}^2$.

Then the relation between (2.2) and (2.3) is given by

$$\begin{aligned} \mathcal{X}_0^+ + \varepsilon \mathcal{X}_1^+ &= J\nabla(H_0^+ + \varepsilon H_1^+), \\ \mathcal{X}_0^- + \varepsilon \mathcal{X}_1^- &= J\nabla(H_0^- + \varepsilon H_1^-), \end{aligned} \tag{2.5}$$

where J is the usual symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We assume that the phase portrait of the unperturbed system (2.2) ($\varepsilon = 0$) is topologically equivalent to the one shown in Figure 1, which we make precise in the following hypotheses.

C.1 There exist two hyperbolic critical points $z^+ \equiv (x^+, y^+) \in S^+$ and $z^- \equiv (x^-, y^-) \in S^-$ of saddle type belonging to the energy level

$$\Big\{(x, y) \,|\, H_0(x, y) = c_1 > 0\Big\}. \tag{2.6}$$

C.2 The form of the Hamiltonian H_0 in (2.4) ensures that $\mathcal{X}_0^{\pm}$ are both tangent to Σ at $(0, 0) \in \Sigma$. We require that $V^{\pm}$ satisfy

$$(V^+)'(0) > 0; (V^-)'(0) < 0,$$

and so $(0, 0)$ is an invisible quadratic tangency for both vector fields. Following [GST11], we call the point $(0, 0)$ an invisible fold-fold.

C.3 There exist two heteroclinic orbits given by $W^u(z^-) = W^s(z^+)$ and $W^u(z^+) = W^s(z^-)$ surrounding the origin and contained in the energy level (2.6).

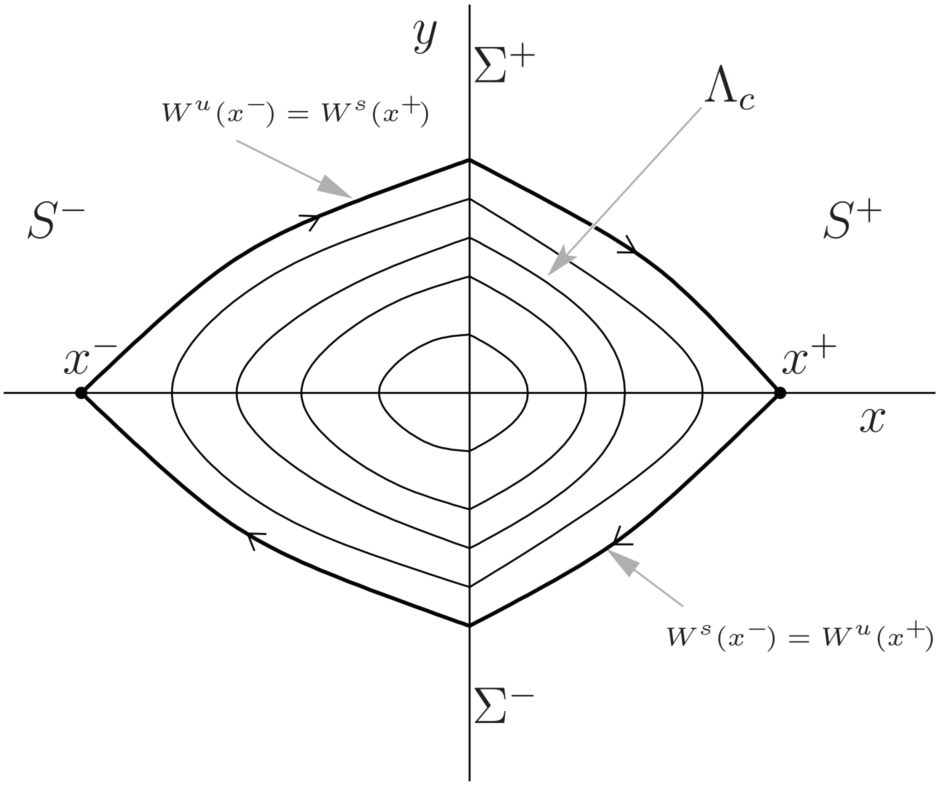


Figure 1. *Phase portrait for the unperturbed system* (2.2).

C.4 The region between both heteroclinic orbits is fully covered by periodic orbits surrounding the origin given by

$$\Lambda_c = \Big\{(x,y) \in \mathbb{R}^2 \,|\, H_0(x,y) = c\Big\} \tag{2.7}$$

with $0 < c < c_1$, and Λ_c intersects Σ transversally exactly twice.

C.5 The period of Λ_c is a regular function of c with strictly positive derivative for $0 < c < c_1$.

Note that, as the unperturbed Hamiltonian H_0 is C^∞ in S^+ and S^-, the fact that the heteroclinic orbits are in the energy level $H_0(x,y) = c_1$ follows automatically from hypothesis C.1. However, we include it explicitly for clarity.

We wish to determine which of these objects and characteristics persist and which are destroyed when the small nonautonomous T-periodic perturbation εH_1 is considered. The splitting of the separatrices and the persistence of periodic orbits is of interest. In the smooth case, these answers are given completely by the classical Melnikov method [GH83]. Hence, it is natural to check whether these classical tools are still valid for the piecewise-smooth system presented above and if any changes to the method are necessary.

Another interesting question that can be addressed with a similar approach is the existence of two-dimensional invariant tori of system (2.2) (see [KKY97, Kun00]).

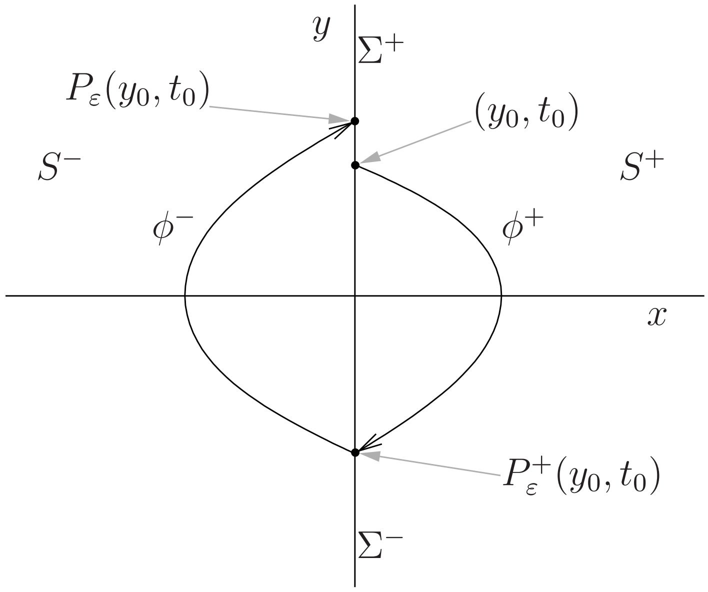


Figure 2. *Poincaré impact map* (2.14) *represented schematically.*

2.2. Poincaré impact map. To study system (2.2) we will proceed as in [Hog89] using the Poincaré impact map. We consider the extended phase space $\mathbb{R}^2\times\mathbb{R}$ adding time as a system variable and equation $\dot{t}=1$ to (2.2). As the perturbation is periodic, this time variable is usually defined in $\mathbb{T}=\mathbb{R}/T$; however, it will be more useful for us to consider $\mathbb{R}$ instead. We want to study the motion in the region surrounded by the heteroclinic orbit, so we consider in this extended phase-space the Poincaré section

$$\tilde{\Sigma}^+=\left\{(0,y,t)\in\mathbb{R}^2\times\mathbb{R}\,|\,0<y<\sqrt{2c_1}\right\}. \tag{2.8}$$

To simplify the notation, as the first coordinate in $\tilde{\Sigma}^+$ is always 0, we will omit its repetition whenever this does not lead to confusion. The domain of the Poincaré map is not $\tilde{\Sigma}^+$ but a suitable open set U that depends on ε and, for $\varepsilon=0$, does not contain the heteroclinic connection.

We now define the Poincaré impact map

$$P_\varepsilon : U\subset\tilde{\Sigma}^+\longrightarrow\tilde{\Sigma}^+,$$

as follows (see Figure 2). First, using the section

$$\tilde{\Sigma}^-=\left\{(0,y,t)\in\mathbb{R}^2\times\mathbb{R}\,|\,-\sqrt{2c_1}<y<0\right\}, \tag{2.9}$$

with $(0,y_0,t_0)\in U^+\subset\tilde{\Sigma}^+$, we define the map

$$P_\varepsilon^+ : U^+\subset\tilde{\Sigma}^+\longrightarrow\tilde{\Sigma}^-,$$

as

$$P_\varepsilon^+(y_0,t_0)=\left(\Pi_y\left(\phi^+\left(t_1;t_0,0,y_0,\varepsilon\right)\right),t_1\right), \tag{2.10}$$

where $\phi^+(t;t_0,x,y,\varepsilon)$ is the flow associated with system (2.2) restricted to S^+, and $t_1 > t_0$ is the smallest value of t satisfying the condition

$$\Pi_x\left(\phi^+\big(t_1;t_0,0,y_0,\varepsilon\big)\right) = 0, \tag{2.11}$$

where Π_x, Π_y are projections onto the x and y axes, respectively.

Similarly, we consider

$$P_\varepsilon^- : U^- \subset \tilde{\Sigma}^- \longrightarrow \tilde{\Sigma}^+$$

for $(0,y_1,t_1) \in U^- \subset \tilde{\Sigma}^-$ defined by

$$P_\varepsilon^-(y_1,t_1) = \big(\Pi_y\left(\phi^-\left(t_2;t_1,0,y_1,\varepsilon\right)\right), t_2\big), \tag{2.12}$$

where $\phi^-(t;t_1,x,y,\varepsilon)$ is the flow associated with (2.2) restricted to S^-, and $t_2 > t_1$ is the smallest value of t satisfying the condition

$$\Pi_x\left(\phi^-\big(t_2;t_1,0,y_1,\varepsilon\big)\right) = 0. \tag{2.13}$$

Then the Poincaré impact map is defined as the composition

$$\begin{array}{rccc} P_\varepsilon : & U \subset \tilde{\Sigma}^+ & \longrightarrow & \tilde{\Sigma}^+ \\ & (y_0,t_0) & \longmapsto & P_\varepsilon^- \circ P_\varepsilon^+(y_0,t_0). \end{array} \tag{2.14}$$

Notice that, as assumed in C.4, for the unperturbed flow all initial conditions in Σ^+ lead to periodic orbits surrounding the origin. Hence, we can give a closed expression for P_0, the Poincaré impact map when $\varepsilon = 0$. Let

$$\alpha^\pm(\pm y) = \pm 2 \int_0^{(V^\pm)^{-1}(h)} \frac{1}{\sqrt{2(h - V^\pm(x))}}\,dx, \quad h = H_0(0,\pm y) = \frac{y^2}{2}, \tag{2.15}$$

be the time needed by an orbit of the unperturbed system with initial condition $(0,\pm y) \in \Sigma^\pm$ to reach $\Sigma^\mp$. In the unperturbed case, the orbit with initial condition $(0,y) \in \Sigma^+$ has period

$$\alpha(y) = \alpha^+(y) + \alpha^-(-y). \tag{2.16}$$

Then the Poincaré impact map when $\varepsilon = 0$ is defined in the whole $\tilde{\Sigma}^+$ and can be written as

$$P_0(y_0,t_0) = (y_0, t_0 + \alpha(y_0)). \tag{2.17}$$

Thus, if ε is small enough, the perturbed trajectories starting at $\tilde{\Sigma}^+$ cross $\tilde{\Sigma}^+$ again. The Poincaré impact map is well defined and is as smooth as the flow restricted to S^+ and S^-.

Note that in the symmetric case, $V^+(x) = V^-(-x)$, $\alpha^+(y) = \alpha^-(-y)$ is half the period of the unperturbed periodic orbit with initial condition $(0,y) \in \Sigma^+$.

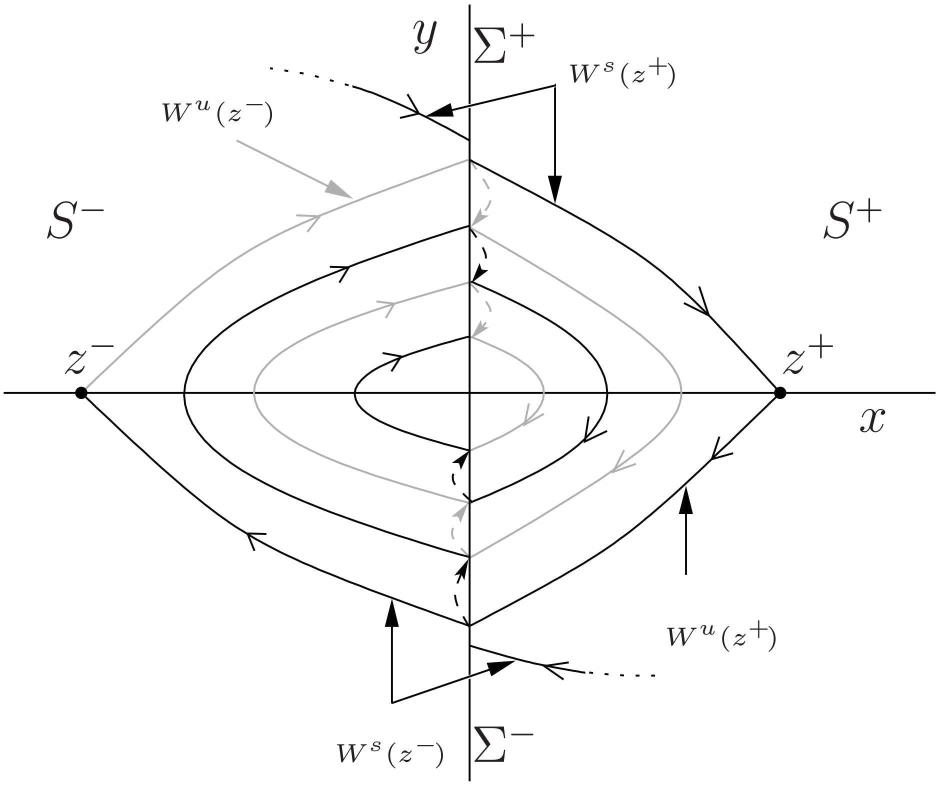


Figure 3. *Stable and unstable manifolds of system* (2.2) *for* $r < 1$ *and* $\varepsilon = 0$.

2.3. Coefficient of restitution. As the name of the previous map suggests, it is typically used to deal with systems with impacts, as is the case of the mechanical example of section 5. In order to include the loss of energy at the impact, one considers a coefficient of restitution, $r \in (0, 1]$, that reduces the velocity, y, at every impact. More precisely, if a trajectory crosses Σ transversally at some point $(0, y_B)$ at $t = t_B$, then the state is replaced by $(0, ry_B)$ at a later time t_A to proceed with the evolution of the system. In other words, the system slides along Σ from $(0, y_B)$ to $(0, ry_B)$ during time $t_A - t_B$ and

$$y(t_A) = ry(t_B). \tag{2.18}$$

For the rest of this article we will assume that the loss of energy is produced instantaneously and hence $t_A = t_B$. Thus, there is no sliding along Σ and the trajectory jumps from $(0, y_B)$ to $(0, ry_B)$.

Clearly, when such a condition is introduced to a system of type (2.2), the unperturbed system ($\varepsilon = 0$) is no longer conservative, the origin becomes a global attractor, and none of the conditions C.1–C.5 holds. In particular, the orbits with initial conditions on the unstable manifolds $W^u(z^-)$ and $W^u(z^+)$ tend to the origin and cannot intersect the stable manifolds $W^s(z^+)$ and $W^s(z^-)$, respectively (see Figure 3).

Although periodic orbits surrounding the origin are not possible for the unperturbed case if $r < 1$, they may exist if $\varepsilon > 0$. However, roughly speaking, as these orbits will have to

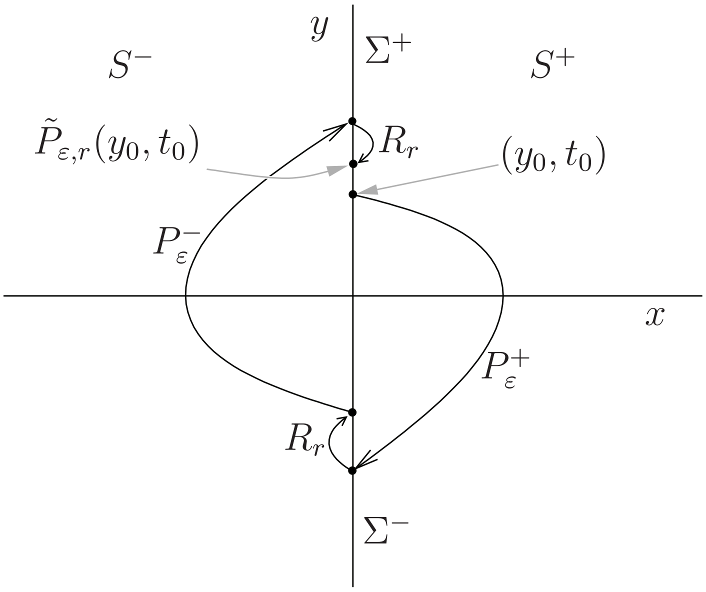


Figure 4. *Impact map for $r<1$ and $\varepsilon>0$.*

overcome the loss of energy, the magnitude of the forcing cannot be arbitrarily small. We will make a precise statement of this fact in section 3.2 (see also [Hog89]).

To study the existence of periodic orbits we will use again the impact map, which can also be defined for $r<1$ as (see Figure 4)

$$\tilde{P}_{\varepsilon,r}(y_0,t_0) := R_r \circ P_\varepsilon^- \circ R_r \circ P_\varepsilon^+(y_0,t_0), \tag{2.19}$$

where

$$R_r(y_0,t_0) = (ry_0,t_0).$$

Note that $\tilde{P}_{\varepsilon,r}$ is as smooth as the flow restricted to $S^\pm$, since it is the composition of smooth maps.

Using (2.15) and (2.16), the impact map, $\tilde{P}_{\varepsilon,r}$, for $\varepsilon=0$ and $r<1$ can be written as

$$\tilde{P}_{0,r}(y_0,t_0) = \left(r^2y_0, t_0+\alpha^+(y_0)+\alpha^-(-ry_0)\right). \tag{2.20}$$

Note that, for any $\varepsilon>0$,

$$\tilde{P}_{\varepsilon,1}(y_0,t_0) = P_\varepsilon(y_0,t_0).$$

2.4. Some formal definitions and notation. Up to now, we have considered separately the solutions of system (2.2) in S^+ and S^- until they reach the switching manifold Σ. Given an initial condition (x_0,y_0,t_0), one can extend the definition of a solution, $\phi(t;t_0,x_0,y_0,\varepsilon,r)$, of system (2.2), (2.18) for all $t\geq t_0$ by properly concatenating ϕ^+ or ϕ^- whenever the flow crosses Σ transversally. Depending on the sign of x_0, one applies either $\phi^+(t;t_0,x_0,y_0,\varepsilon)$ or $\phi^-(t;t_0,x_0,y_0,\varepsilon)$ until the trajectory reaches Σ, and then one applies (2.18). If $x_0=0$, one proceeds similarly depending on the sign of y_0. This is because $\dot{x}=y+O(\varepsilon)$ is always an equation of the flow and the orbits twist clockwise.

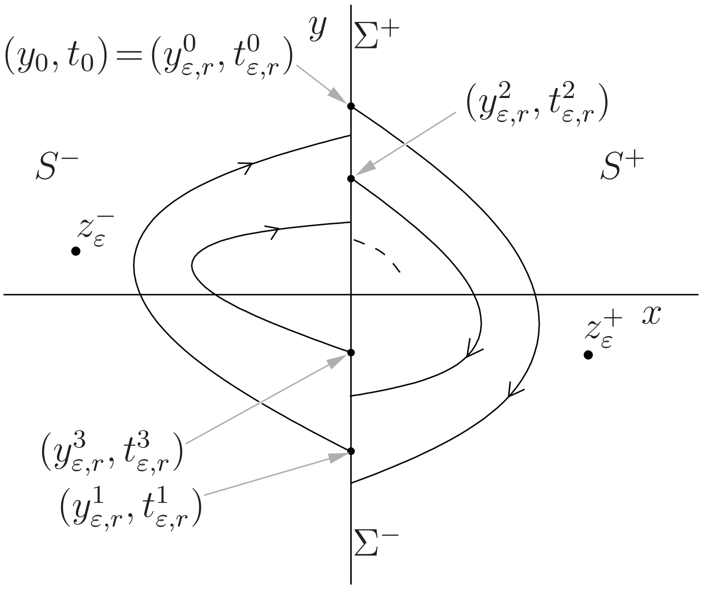


Figure 5. *Sequence of impacts for $r < 1$ and $\varepsilon > 0$.*

In this work, we will mainly use solutions with initial conditions $(0, y_0, t_0) \in \tilde{\Sigma}^+$. In that case, we define the sequence of impacts $(0, y^i_{\varepsilon,r}, t^i_{\varepsilon,r})$ (see Figure 5), if they exist, as

$$(y^i_{\varepsilon,r}, t^i_{\varepsilon,r}) = \begin{cases} R_r \circ P^-_\varepsilon(y^{i-1}_{\varepsilon,r}, t^{i-1}_{\varepsilon,r}) & \text{if } y^{i-1}_{\varepsilon,r} < 0, \\ R_r \circ P^+_\varepsilon(y^{i-1}_{\varepsilon,r}, t^{i-1}_{\varepsilon,r}) & \text{if } y^{i-1}_{\varepsilon,r} > 0, \end{cases} \tag{2.21}$$

with $(y^0_{\varepsilon,r}, t^0_{\varepsilon,r}) = (y_0, t_0)$ and $P^\pm_\varepsilon$ defined in (2.10) and (2.12). Notice that the sequence (2.21) will be finite if the flow reaches Σ a finite number of times only.

For the unperturbed case, for any point $(0, y_0, t_0) \in \tilde{\Sigma}^+$, the sequence (2.21) becomes

$$(y^i_{0,r}, t^i_{0,r}) := \begin{cases} \left(r^i y_0, t^{i-1}_{0,r} + \alpha^-\left(-r^{i-1} y_0\right)\right) & \text{if } i \geq 2 \text{ even}, \\ \left(-r^i y_0, t^{i-1}_{0,r} + \alpha^+\left(r^{i-1} y_0\right)\right) & \text{if } i \geq 1 \text{ odd}, \end{cases} \tag{2.22}$$

where $\alpha^\pm$ are defined in (2.15).

Once the impacts $(y^i_{\varepsilon,r}, t^i_{\varepsilon,r})$ are defined, the solution of the nonautonomous system (2.2), (2.18) with initial condition $(0, y_0, t_0) \in \tilde{\Sigma}^+$ is given by

$$\phi(t; t_0, 0, y_0, \varepsilon, r) := \begin{cases} \phi^+(t; t^{2i}_{\varepsilon,r}, 0, y^{2i}_{\varepsilon,r}, \varepsilon) & \text{if } t^{2i}_{\varepsilon,r} \leq t < t^{2i+1}_{\varepsilon,r}, \\ \\ \phi^-(t; t^{2i+1}_{\varepsilon,r}, 0, y^{2i+1}_{\varepsilon,r}, \varepsilon) & \text{if } t^{2i+1}_{\varepsilon,r} \leq t < t^{2i+2}_{\varepsilon,r}, \end{cases} \quad i \geq 0. \tag{2.23}$$

Note that in the case when the number of impacts is finite, we take the last interval of time to be infinitely long.

In the rest of the paper we will generally distinguish between the conservative ($r = 1$) and dissipative ($r < 1$) cases. We will omit the parameter r in the flow ϕ whenever we refer to $r = 1$.

Note that we have only defined the solution of the system for an initial condition $(0, y_0, t_0) \in \tilde{\Sigma}^+$. Given $(0, y_0, t_0) \in \tilde{\Sigma}^-$, one defines similarly this solution by just properly shifting the subscripts of t_ε^i in (2.23). In addition, it is possible to extend precisely this definition to an arbitrarily initial condition (x_0, y_0, t_0).

As is usual when dealing with Hamiltonian systems, we will use the unperturbed Hamiltonian to measure the distance between states. In addition, as we are dealing with a perturbation problem, we will frequently use expansions in powers of ε. In this case, the integral of the Poisson brackets of the Hamiltonians H_1 and H_0 typically provides a compact expression for the linear terms in ε. Given $m \geq 1$, $(0, y_0, t_0) \in \tilde{\Sigma}^+$, and its impact sequence $(0, y^i_{\varepsilon,r}, t^i_{\varepsilon,r})$, $0 \leq i \leq 2m$, for the piecewise-smooth system (2.2), (2.18) when $r \leq 1$, we introduce

$$
\begin{aligned}
&\int_{t_0}^{t^{2m}_{\varepsilon,r}} \{H_0, H_1\}\left(\phi\left(t; t_0, 0, y_0, \varepsilon, r\right), t\right) dt \\
&\qquad := \sum_{i=0}^{m-1} \left(\int_{t^{2i}_{\varepsilon,r}}^{t^{2i+1}_{\varepsilon,r}} \left\{H_0^+, H_1^+\right\}\left(\phi^+(t; t^{2i}_{\varepsilon,r}, 0, y^{2i}_{\varepsilon,r}, \varepsilon), t\right) dt \right. \\
&\qquad\quad \left. + \int_{t^{2i+1}_{\varepsilon,r}}^{t^{2i+2}_{\varepsilon,r}} \left\{H_0^-, H_1^-\right\}\left(\phi^-(t; t^{2i+1}_{\varepsilon,r}, 0, y^{2i+1}_{\varepsilon,r}, \varepsilon), t\right) dt \right),
\end{aligned}
\tag{2.24}
$$

where $\{Q(x,y), R(x,y)\} = \frac{\partial Q}{\partial x}\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial y}\frac{\partial R}{\partial x}$ is the canonical Poisson bracket of the Hamiltonians Q and R.

The next lemma provides an expression for $H_0\left(\phi\left(t^{2m}_{\varepsilon,r}; t_0, 0, y_0, \varepsilon, r\right)\right)$, which we will use below.

Lemma 2.1. *Let $m \geq 1$ and $(0, y_0, t_0) \in \tilde{\Sigma}^+$, and let $(0, y^i_{\varepsilon,r}, t^i_{\varepsilon,r})$, $i = 0, \ldots, 2m$, be its associated impact sequence as defined in* (2.21). *Then*

$$
\begin{aligned}
H_0(0, y^{2m}_{\varepsilon,r}) - H_0(0, y_0) = r^2 &\left[\varepsilon \int_{t_0}^{t^{2m}_{\varepsilon,r}} \{H_0, H_1\}\left(\phi\left(t; t_0, 0, y_0, \varepsilon, r\right), t\right) dt \right. \\
&\left. + \sum_{i=0}^{2m-1} \left(H_0\left(0, y^i_{\varepsilon,r}\right) - H_0\left(0, \frac{y^i_{\varepsilon,r}}{r}\right)\right)\right].
\end{aligned}
\tag{2.25}
$$

Proof. The proof of this lemma comes from a straightforward application of the fundamental theorem of calculus to the smooth functions $H_0^\pm\left(\phi^\pm\left(t; t_0, 0, \pm y_0, \varepsilon\right)\right)$, using the fact that

$$
\begin{aligned}
H_0(0, y^{2m}_{\varepsilon,r}) &= H_0\left(r\phi\left(t^{2m}_{\varepsilon,r}; t_0, 0, y_0, \varepsilon, r\right)\right) \\
&= r^2 H_0\left(\phi\left(t^{2m}_{\varepsilon,r}; t_0, 0, y_0, \varepsilon, r\right)\right),
\end{aligned}
$$

taking into account the intermediate gaps induced by the impact condition (2.18) and that

$$
\frac{d}{dt} H_0^\pm\left(\phi^\pm\left(t; t^*, x^*, y^*, \varepsilon\right)\right) = \varepsilon \left\{H_0^\pm, H_1^\pm\right\}\left(\phi^\pm\left(t; t^*, x^*, y^*, \varepsilon\right)\right)
$$

for any $(x^*, y^*) \in S^\pm \cup \Sigma^\pm$ and $t \geq t^*$ such that $\phi^\pm\left(t; t^*, x^*, y^*, \varepsilon\right) \in S^\pm$. ∎

The following lemma gives us an expression for the expansion in powers of ε of $H_0(0, y_{\varepsilon,r}^{2m}) - H_0(0, y_0)$, which we will use in section 3.

Lemma 2.2. *Let $m \geq 1$ and $(0, y_0, t_0) \in \tilde{\Sigma}^+$, and let $(0, y_{\varepsilon,r}^i, t_{\varepsilon,r}^i)$, $i = 0, \dots, 2m$, be its associated impact sequence as defined in (2.21). Then, if $\varepsilon \simeq 0$, the Taylor expansion of expression (2.25) becomes*

$$H_0(0, y_{\varepsilon,r}^{2m}) - H_0(0, y_0) = \frac{y_0^2}{2}(r^{4m} - 1) + \varepsilon G^m(y_0, t_0) + O(\varepsilon^2) + O(\varepsilon (r-1)), \tag{2.26}$$

where

$$G^m(y_0, t_0) = \int_0^{m\alpha(y_0)} \{H_0, H_1\} (\phi(t; 0, 0, y_0, 0), t + t_0)\, dt \tag{2.27}$$

and $\alpha(y_0)$ is given in (2.16).

Proof. The zero order term in ε of the expansion is found by noting that if $\varepsilon = 0$, from expression (2.22), one has $H_0(0, y_{0,r}^i) = H_0(0, \frac{y_{0,r}^{i+1}}{r})$. Hence all the terms in the sum of (2.25) cancel each other except for the first and the last one. This, in combination with the fact that $H_0(0, y) = \frac{y^2}{2}$, gives the first term in (2.26). For the linear term in ε, one first obtains

$$r^2 \left[\int_{t_0}^{t_{\varepsilon,r}^{2m}} \{H_0, H_1\} (\phi(t; t_0, 0, y_0, 0, r), t)\, dt + (r^2 - 1) \sum_{i=1}^{2m-1} \left(\frac{d}{d\varepsilon} \left(H_0\left((y_{\varepsilon,r}^i, t_{\varepsilon,r}^i)\right)\right)_{|\varepsilon=0} \right) \right].$$

Then, by applying (2.20) m times, one has that $t_{\varepsilon,1}^{2m} = t_0 + m\alpha(y_0) + O(\varepsilon)$. Thus, by expanding this for r near 1 and ε near 0 and noting that the unperturbed flow is autonomous and hence $\phi(t; t_0, 0, y_0, 0) = \phi(t - t_0; 0, 0, y_0, 0)$, one gets expression (2.27). ∎

Remark 2.1. If in (2.27) we take $\alpha(y_0) = \frac{nT}{m}$, then we recover the classical Melnikov function for the subharmonic orbits [GH83] with the modified integral (2.24).

3. Existence of subharmonic orbits.

3.1. Conservative case, $r = 1$: Melnikov method for subharmonic orbits. Let us consider system (2.2) neglecting the loss of energy at impact ($r = 1$ in (2.18)). According to C.1–C.5, for $\varepsilon = 0$, this system possesses a continuum of periodic orbits, Λ_c in (2.7), surrounding the origin. Our main goal in this section is to investigate the persistence of these orbits when the (periodic) nonautonomous perturbation is considered ($\varepsilon > 0$). The classical Melnikov method for subharmonic orbits, which here, in principle, does not apply, provides sufficient conditions for the persistence of periodic orbits for a smooth system with an equivalent, smooth, unperturbed phase portrait.

The period of the orbits Λ_c tends to infinity as they approach the heteroclinic orbit. More precisely, if $q_c(t)$ is the periodic orbit satisfying $q_c(0) = (0, y_0)$ with $H_0(0, y_0) = c$, its period $\alpha(y_0)$ tends to infinity as $c \to c_1$ (see formula (2.16)). As we are interested in finding periodic orbits for $0 < \varepsilon \ll 1$, we will use the unperturbed periodic solutions as ε-close approximations to them. In general, such perturbation results are valid only for finite time and therefore, from now on, we will restrict ourselves to a set of the form

$$\tilde{\Sigma}^+_{\tilde{c}} = \left\{(0, y, t) \in \tilde{\Sigma}^+ \,|\, 0 \leq y \leq \tilde{c}\right\} \tag{3.1}$$

for a fixed $\tilde{c}$ satisfying $0 \leq \tilde{c} < \sqrt{2c_1}$. Note that if $(0, y_0, t_0) \in \tilde{\Sigma}^+_{\tilde{c}}$, then $\alpha(y_0)$ is uniformly bounded ($\alpha(y_0) < \alpha(\tilde{c})$). However, following [GH83], it is also possible to extend the method for all the periodic orbits up to the heteroclinic connection.

To look for periodic orbits we will use the impact map defined in (2.14). In terms of this map, a point in $U \subset \tilde{\Sigma}^+$ will lead to a periodic orbit of period nT if it is a solution of the equation

$$P^m_\varepsilon(y_0, t_0) = (y_0, t_0 + nT) \tag{3.2}$$

for some m. We take m to be the smallest integer such that (3.2) is satisfied. In that case, $\phi\,(t; t_0, 0, y_0, \varepsilon)$ will be a periodic orbit of period nT, which crosses the switching manifold Σ exactly $2m$ times. We will call this an (n, m)-periodic orbit. Then for (n, m)-periodic orbits with $\varepsilon > 0$ we have the following result analogous to the smooth case.

Theorem 3.1. *Consider a system as defined in* (2.2) *satisfying* C.1–C.5*, and let* $\alpha(y_0)$ *be the function defined in* (2.15)–(2.16). *Assume that the point* $(0, \bar{y}_0, \bar{t}_0) \in \tilde{\Sigma}^+_{\tilde{c}}$ *satisfies the following.*

H.1 $\alpha(\bar{y}_0) = \frac{n}{m}T$*, with* $n, m \in \mathbb{Z}$ *relatively prime.*

H.2 $\bar{t}_0 \in [0, T]$ *is a simple zero of*

$$M^{n,m}(t_0) = \int_0^{nT} \{H_0, H_1\}\,(q_c(t), t + t_0)dt, \; c = H_0(0, \bar{y}_0), \tag{3.3}$$

where $q_c(t) = \phi\,(t; 0, 0, \bar{y}_0)$ *is the periodic orbit such that* $\alpha(\bar{y}_0) = \frac{nT}{m}$.

Then there exists ε_0 *such that, for every* $0 < \varepsilon < \varepsilon_0$*, one can find* y_0^* *and* t_0^* *such that* $\phi(t; t_0^*, 0, y_0^*, \varepsilon)$ *is an* (n, m)*-periodic orbit.*

Proof. The proof of the result comes from a straightforward application of the implicit function theorem to (3.2). Let us fix n and m relatively prime. We replace (3.2) by

$$\begin{pmatrix} H_0\left(0, \Pi_{y_0}(P^m_\varepsilon(y_0, t_0))\right) \\ \Pi_{t_0}\left(P^m_\varepsilon(y_0, t_0)\right) \end{pmatrix} - \begin{pmatrix} H_0(0, y_0) \\ t_0 + nT \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

That is, we use the Hamiltonian H_0 to measure the distance between the points $(0, \Pi_{y_0}\left(P^m_\varepsilon\left(y_0, t_0\right)\right))$ and $(0, y_0)$.

Using the second equation in (3.4) we have

$$\begin{aligned} \Pi_{y_0}\left(P^m_\varepsilon\left(y_0, t_0\right)\right) &= \Pi_y\left(\phi(t_0 + nT; t_0, 0, y_0, \varepsilon)\right), \\ 0 &= \Pi_x\left(\phi\left(t_0 + nT; t_0, 0, y_0, \varepsilon\right)\right). \end{aligned}$$

This allows us to rewrite (3.4) as

$$\begin{pmatrix} H_0(\phi(t_0 + nT; t_0, 0, y_0, \varepsilon)) - H_0(0, y_0) \\ \Pi_{t_0}\left(P^m_\varepsilon\left(y_0, t_0\right)\right) - nT - t_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

We expand (3.5) in powers of ε. Using (2.17), the second component of (3.5) becomes

$$\Pi_{t_0}\left(P_\varepsilon^m\left(y_0,t_0\right)\right)-t_0-nT=m\alpha(y_0)-nT+O(\varepsilon)=0, \tag{3.6}$$

where $\alpha(y_0)$ is the period of the periodic orbit $q_c(t)$, $c=H_0(0,y_0)$, given in (2.16).

On the other hand, using Lemma 2.2 and noting that

$$\Pi_{y_0}\left(P_\varepsilon^m\left(y_0,t_0\right)\right)=y_{\varepsilon,1}^{2m},$$

the first equation in (3.5) can be written as

$$\begin{aligned}
H_0(0,\Pi_{y_0}(P_\varepsilon^m(y_0,t_0)))&-H_0(0,y_0)\\
&=\varepsilon\int_0^{m\alpha(y_0)}\{H_0,H_1\}\left(\phi(t;0,0,y_0,0),t+t_0\right)dt+O(\varepsilon^2)\\
&=\varepsilon G^m(y_0,t_0)+O(\varepsilon^2),
\end{aligned}$$

where $G^m(y_0,t_0)$ is given in (2.27). Hence, (3.5) finally becomes

$$F_{n,m}(y_0,t_0,\varepsilon):=\begin{pmatrix} G^m(y_0,t_0)+O(\varepsilon)\\ m\alpha(y_0)-nT+O(\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{3.7}$$

where the order in ε of the first component has been reduced and, thus, the implicit function theorem can be applied to (3.7). Therefore, one needs

1. $F_{n,m}(\bar{y}_0,\bar{t}_0,0)=(0,0)^T$;
2. $\det(D_{y_0,t_0}F(\bar{y}_0,\bar{t}_0,0))\neq 0$, where $D_{y_0,t_0}\equiv D$ is the Jacobian with respect to the variables y_0 and t_0.

The first condition is satisfied by noting in (3.7) that $\bar{y}_0$ has to be such that $\alpha(\bar{y}_0)=\frac{nT}{m}$ and $\bar{t}_0$ is a zero of the subharmonic Melnikov function

$$M^{n,m}(t_0):=G^m(\bar{y}_0,t_0)=\int_0^{nT}\{H_0,H_1\}\left(q_c(t),t+t_0\right)dt,$$

where $q_c(t)$, $c=H_0(0,\bar{y}_0)$, is the unperturbed periodic orbit of period $\frac{nT}{m}$ such that $q_c(0)=(0,y_0)$, and therefore $q_c(t)=\phi\left(t;0,0,\bar{y}_0,0\right)$.

In addition, for $\varepsilon=0$, $DF_{n,m}$ is given by

$$DF_{n,m}(y_0,t_0,0)=\begin{pmatrix}\frac{\partial G^m}{\partial y_0} & \frac{\partial G^m}{\partial t_0}\\ m\alpha'(y_0) & 0\end{pmatrix}.$$

By C.5, $\alpha'(y_0)\neq 0$, and the second condition is satisfied if $\bar{t}_0$ is a simple zero of the subharmonic Melnikov function, $M^{n,m}(t_0)$, which completes hypothesis H.2.

Finally, applying the implicit function theorem to (3.7) at $(y_0,t_0,\varepsilon)=(\bar{y}_0,\bar{t}_0,0)$, there exists $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$, then there exist unique $y_0^*(\varepsilon)$ and $t_0^*(\varepsilon)$ solutions of (3.4), which have the form

$$\begin{aligned}
y_0^*&=\bar{y}_0+O(\varepsilon),\\
t_0^*&=\bar{t}_0+O(\varepsilon).
\end{aligned}$$

Hence, the orbit $\phi(t;t_0^*,0,y_0^*,\varepsilon)$ is an (n,m)-periodic orbit, as it has period nT and impacts $2m$ times with the switching manifold Σ in every period. ■

Remark 3.1. The upper bound ε_0 given in the theorem depends on n and m. However, for every fixed m, it is possible to obtain $\varepsilon_0(m)$, such that for $\varepsilon < \varepsilon_0(m)$, we can apply the theorem for all n such that $\alpha^{-1}(\frac{nT}{m}) \in \tilde{\Sigma}_{\bar{c}}^{+}$. This is because the approximation of the perturbed flow by the unperturbed periodic orbit is performed m times beyond the period of the unperturbed periodic orbit.

Remark 3.2. The proof of the result provides us with a constructive method for finding the initial condition for nT-periodic orbits for $\varepsilon > 0$.

1. Given n and m, find $\bar{y}_0$ such that $\alpha(\bar{y}_0) = \frac{n}{m}T$ using (2.16).
2. Find $\bar{t}_0$ such that $M^{n,m}(t_0)$ has a simple zero at $t_0 = \bar{t}_0$.
3. Use $(\bar{y}_0, \bar{t}_0)$ as a seed to solve (3.4) numerically.

Lemma 3.2. *The subharmonic Melnikov function* (3.3) *is either identically zero or generically possesses at least one simple zero.*

Proof. The proof comes from the fact that $M^{n,m}(t_0)$ has average

$$\langle M^{n,m}(t_0)\rangle = \frac{1}{T}\int_0^T M^{n,m}(t_0)dt_0$$

equal to zero.

$$\begin{aligned}\langle M^{n,m}(t_0)\rangle &= \frac{1}{T}\int_0^T\int_0^{nT}\{H_0,H_1\}(q_c(t),t+t_0)\,dtdt_0\\ &= \frac{1}{T}\int_0^{nT}\int_0^{T}\{H_0,H_1\}(q_c(t),t+t_0)\,dt_0dt\\ &= \int_0^{nT}\{H_0,\langle H_1\rangle\}(q_c(t))dt.\end{aligned}$$

Recalling that $\alpha(y_0) = \frac{nT}{m}$ (see (2.15)–(2.16)) and letting

$$\begin{aligned}q_c^+(t) &= \phi^+(t;0,0,y_0,0),\\ q_c^-(t) &= \phi^-(t;\alpha^+(y_0),0,-y_0,0),\end{aligned}$$

$\langle M^{n,m}(t_0)\rangle$ can be written as

$$\begin{aligned}&m\left(\int_0^{\alpha^+(y_0)}\{H_0^+,\langle H_1^+\rangle\}\left(q_c^+(t)\right)dt + \int_{\alpha^+(y_0)}^{\frac{nT}{m}}\{H_0^-,\langle H_1^-\rangle\}\left(q_c^-(t)\right)dt\right)\\ &= -m\left(\int_0^{\alpha^+(y_0)}\frac{d}{dt}\left(\langle H_1^+\rangle\left(q_c^+(t)\right)\right)dt + \int_{\alpha^+(y_0)}^{\frac{nT}{m}}\frac{d}{dt}\left(\langle H_1^-\rangle\left(q_c^-(t)\right)\right)dt\right)\\ &= -m\Bigg(\langle H_1^+\rangle\left(q_c^+(\alpha^+(y_0))\right) - \langle H_1^+\rangle\left(q_c^+(0)\right)\\ &\quad + \langle H_1^-\rangle\left(q_c^-(\tfrac{nT}{m})\right) - \langle H_1^-\rangle\left(q_c^-\left(\alpha^+(y_0)\right)\right)\Bigg) = 0.\end{aligned}$$

■

Note that if $M^{n,m}(t_0) \equiv 0$, then a second order analysis is required to study the existence of periodic orbits.

3.2. Dissipative case, $r < 1$. We now focus on the situation when the coefficient of restitution r introduced in section 2.3 is considered. As already mentioned, for $\varepsilon = 0$ the origin is a global attractor, and hence none of the periodic orbits studied in the previous section exists if the amplitude of the perturbation is small enough. However, as was shown in [Hog89] for the rocking block model, for ε large enough an infinite number of periodic orbits surrounding the origin can exist. This was studied analytically and numerically for the rocking block model under symmetry assumptions for the particular case $m = 1$. Here, our goal is to relate the periodic orbits existing for the dissipative case to those which exist for $r = 1$ in the general system (2.2), (2.18). As will be shown below, all the periodic orbits given by Theorem 3.1 can also exist for the dissipative case, when $r < 1$ is small enough compared with $\varepsilon > 0$. In other words, we generalize in this section the result presented for the conservative case.

As in section 3.1, in order to obtain the initial conditions of an (n,m)-periodic orbit for $r < 1$, one has to solve the equation

$$\tilde{P}^m_{\varepsilon,r}(y_0, t_0) = (y_0, t_0 + nT), \tag{3.8}$$

where $\tilde{P}_{\varepsilon,r}$ is defined in (2.19). The next result states that, under certain conditions regarding r and ε, (3.8) can be solved.

Theorem 3.3. *Consider system (2.2), (2.18). Let $(0, \bar{y}_0, \bar{t}_0) \in \tilde{\Sigma}^+$ be such that $\alpha(\bar{y}_0) = \frac{nT}{m}$, with n and m relatively prime, and $\bar{t}_0$ a simple zero of the subharmonic Melnikov function (3.3). There exists ρ such that, given $\tilde{\varepsilon}, \tilde{r} > 0$ satisfying $0 < \frac{\tilde{r}}{\tilde{\varepsilon}} < \rho$, there exists δ_0 such that if $\varepsilon = \tilde{\varepsilon}\delta$ and $r = 1 - \tilde{r}\delta$, then for all $0 < \delta < \delta_0$ there exists (y_0^*, t_0^*) which is a solution of (3.8). Moreover, $y_0^* = \bar{y}_0 + O(\delta)$, $t_0^* = \bar{t}_0 + O(\delta) + O(\tilde{r}/\tilde{\varepsilon})$, and the solution (y_0^*, t_0^*) tends to the one given in Theorem 3.1 as $r \to 1^-$.*

Proof. As in the conservative case, we use the unperturbed Hamiltonian to measure the distance between points in Σ. Then (3.8) can be rewritten as

$$\begin{pmatrix} H_0\left(0, \Pi_{y_0}(\tilde{P}^m_{\varepsilon}(y_0, t_0))\right) \\ \Pi_{t_0}\left(\tilde{P}^m_{\varepsilon}(y_0, t_0)\right) \end{pmatrix} - \begin{pmatrix} H_0(0, y_0) \\ t_0 + nT \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.9}$$

As in Theorem 3.1, we proceed by expanding this equation in powers of ε and $r-1$ using (2.26) and (2.27), obtaining

$$\begin{pmatrix} \frac{y_0^2}{2}(r^{4m} - 1) + \varepsilon G^m(y_0, t_0) + O(\varepsilon^2) + O(\varepsilon(r-1)) \\ \displaystyle\sum_{i=0}^{m-1} \alpha^+(r^{2i} y_0) + \sum_{i=0}^{m-1} \alpha^-\left(-r^{2i+1} y_0\right) + O(\varepsilon) - nT \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

Note that, for $r = 1$, (3.10) becomes (3.5).

We are interested in studying (3.10) when $1 - r$ and ε are both small. Therefore, for $\tilde{\varepsilon} > 0$ and $\tilde{r} > 0$ we set

$$\varepsilon = \tilde{\varepsilon}\delta, \; r = 1 - \tilde{r}\delta, \tag{3.11}$$

where $\delta > 0$ is a small parameter. Then (3.10) becomes

$$\tilde{F}_{n,m}(y_0, t_0, \delta) := \begin{pmatrix} -2m\tilde{r}y_0^2 + \tilde{\varepsilon}G^m(y_0, t_0) + O(\delta) \\ m\alpha(y_0) + O(\delta) - nT \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.12}$$

We now need to apply the implicit function theorem to (3.12).

The first step is to solve (3.12) for $\delta = 0$. The second equation gives $\alpha(\bar{y}_0) = \frac{nT}{m}$, as in Theorem 3.1. To solve the first equation, we define

$$f^{n,m}(t_0) = -2m\tilde{r}\bar{y}_0^2 + \tilde{\varepsilon}M^{n,m}(t_0), \tag{3.13}$$

and $\widehat{t_0}$ will be given by a zero of $f^{n,m}(t_0)$. Assume now that $\bar{t}_0$ is a simple zero of $M^{n,m}(t_0)$. As $M^{n,m}(t_0)$ is a smooth periodic function, it possesses at least one local maximum. Let t_M be the closest value to $\bar{t}_0$ where $M^{n,m}(t_0)$ possesses a local maximum, and assume $(M^{n,m})'(t_0) \neq 0$ for all t_0 between $\bar{t}_0$ and t_M. If $(M^{n,m})'(t_0)$ vanishes between $\bar{t}_0$ and t_M, we then take t_M to be the closest value to $\bar{t}_0$ such that $(M^{n,m})'(t_0) = 0$ to ensure that $(M^{n,m})'(t_0) \neq 0$ between $\bar{t}_0$ and t_M. We then define $\rho := \frac{M^{n,m}(t_M)}{2m\bar{y}_0^2}$. Then if

$$0 < \frac{\tilde{r}}{\tilde{\varepsilon}} < \rho, \tag{3.14}$$

there exists $\widehat{t_0}$ $\frac{\tilde{r}}{\tilde{\varepsilon}}$-close to $\bar{t}_0$ where $f^{n,m}(t_0)$ has a simple zero. Since $\alpha'(\bar{y}_0) > 0$, a similar calculation to the one in Theorem 3.1 shows that

$$\det\left(D\tilde{F}_{y_0,t_0}\left(\bar{y}_0, \widehat{t_0}, 0\right)\right) \neq 0,$$

and hence we can apply the implicit function theorem near $(y_0, t_0, \delta) = (\bar{y}_0, \widehat{t_0}, 0)$ to show that there exists δ_0 such that if $0 < \delta < \delta_0$, then there exists

$$(y_0^*, t_0^*) = (\bar{y}_0, \widehat{t_0}) + O(\delta) = (\bar{y}_0, \bar{t}_0) + O(\delta) + O(\tilde{r}/\tilde{\varepsilon}),$$

which is a solution of (3.8).

This solution tends to the one given by Theorem 3.1 when $\tilde{r} \to 0^+$. This is a natural consequence of the fact that $\tilde{P}^m_{\varepsilon,r}$ uniformly tends to P^m_ε as $r \to 1^-$. ■

Remark 3.3. In order to determine ρ in (3.14), we have imposed t_M to be the local maximum of the Melnikov function closest to its simple zero, $\bar{t}_0$. Instead, one could also use the absolute maximum, thus increasing the range given in (3.14). However, in this case, the values where $(M_1^{n,m})'(t_0) = 0$ have to be avoided to ensure that the desired zero of $f^{n,m}(t_0)$ is simple.

Remark 3.4. Arguing as in Remark 3.1, for every m fixed, the constant $\delta_0(m)$ can be taken such that if $\delta < \delta_0(m)$, there exist periodic orbits for all n such that $\alpha(\frac{nT}{m})^{-1} \in \tilde{\Sigma}$.

4. Intersection of the separatrices. We now focus our attention on the invariant manifolds of the saddle points of system (2.2), (2.18) when $\varepsilon > 0$. As explained in section 2, for $\varepsilon = 0$, there exist two heteroclinic orbits connecting the critical points $z^\pm$ if $r = 1$ (see Figure 1), whereas if $r < 1$, the unstable manifolds $W^u(z^\pm)$ spiral discontinuously from $z^\pm$ to the origin and $W^s(z^\pm)$ becomes unbounded (see Figure 3). As we will show, in both cases heteroclinic orbits may exist for the perturbed system.

For a smooth system with Hamiltonian $K_0(x,y)+\varepsilon K_1(x,y,t)$, the persistence of homoclinic or heteroclinic connections is achieved by the well-known Melnikov method, which states that if the Melnikov function

$$M(t_0)=\int_{-\infty}^{+\infty}\left\{K_0,K_1\right\}\left(\phi\left(t;t_0,z_0,0\right),t+t_0\right)dt,$$

with $z_0=(x_0,y_0)\in W^u(z^-)=W^s(z^+)$, has a simple zero, then the stable and unstable manifolds intersect for $\varepsilon>0$ small enough (see [GH83]).

In this section we will modify the classical Melnikov method, and we will rigorously prove that it is still valid for a piecewise-smooth system of the form (2.2), even if $r\leq 1$.

There exist in the literature several works where this tool has been used in particular piecewise-smooth examples [Hog92, BK91]. Theorem 4.1 generalizes the result stated in [BK91] where the Melnikov method is shown to work, although the proof there is not complete.

The homoclinic version of a piecewise-defined system with a different topology was studied in [Kun00, Kuk07, BF08, DZ05, XFR09]. However, as noted in the introduction, the tools developed there do not apply for a system of the type (2.2).

We begin by discussing the persistence of objects for $\varepsilon>0$ and $r\leq 1$. It is clear that by separately extending the systems $\mathcal{X}_0^{\pm}+\varepsilon\mathcal{X}_1^{\pm}$ to $\mathbb{R}^2\times\mathbb{T}$, where $\mathbb{T}=\mathbb{R}/T$, we get two smooth systems for which the classical perturbation theory holds. It follows then that, as $z^{\pm}$ are hyperbolic fixed points, for $\varepsilon>0$ small enough there exist two hyperbolic T-periodic orbits, $\Lambda_\varepsilon^{\pm}\equiv\{z_\varepsilon^{\pm}(\tau);\,\tau\in[0,T]\}$, with two-dimensional stable and unstable manifolds $W^{s,u}(\Lambda_\varepsilon^{\pm})$.

As the system is nonautonomous, we fix the Poincaré section

$$\Theta_{t_0}=\left\{(x,y,t_0),\,(x,y)\in\mathbb{R}^2\right\}$$

and consider the time T stroboscopic map

$$\Pi_{t_0}:\Theta_{t_0}\longrightarrow\Theta_{t_0+T},$$

where

$$\Pi_{t_0}(z)=\phi(t_0+T;t_0,z,\varepsilon,r)$$

and ϕ is as defined section 2.4.

This map has $z_\varepsilon^{\pm}(t_0)$ as hyperbolic fixed points with one-dimensional stable and unstable manifolds (curves) $W^{s,u}(z_\varepsilon^{\pm}(t_0))$ (see Figure 3). Proceeding as in [BK91], we fix the section Σ defined in (2.1) and study its intersection with the stable and unstable manifolds $W^u(z_\varepsilon^-(t_0))$ and $W^s(z_\varepsilon^+(t_0))$. In the unperturbed conservative case ($\varepsilon=0$ and $r=1$), $W^u(z^-)$ and $W^s(z^+)$ intersect Σ transversally in a point z_0. The perturbed manifolds, $W^u(z_\varepsilon^-(t_0))$ and $W^s(z_\varepsilon^+(t_0))$, intersect Σ at points $z^u(t_0)$ and $z^s(t_0)$, respectively, ε-close to z_0 (see Figure 6). Recalling the effect of the coefficient of restitution (2.18) explained in section 2.3, both invariant manifolds will intersect if, for some t_0, one has $rz^u(t_0)=z^s(t_0)$, $r\leq 1$. As in [BK91] and [Hog92], we use the unperturbed Hamiltonian $H_0(x,y)$ to measure the distance $\Delta(t_0,\varepsilon,r)$ between z^u and z^s,

$$\Delta(t_0,\varepsilon,r)=H_0(rz^u(t_0))-H_0(z^s(t_0))=r^2H_0(z^u(t_0))-H_0(z^s(t_0)).\tag{4.1}$$

We then have the following result.

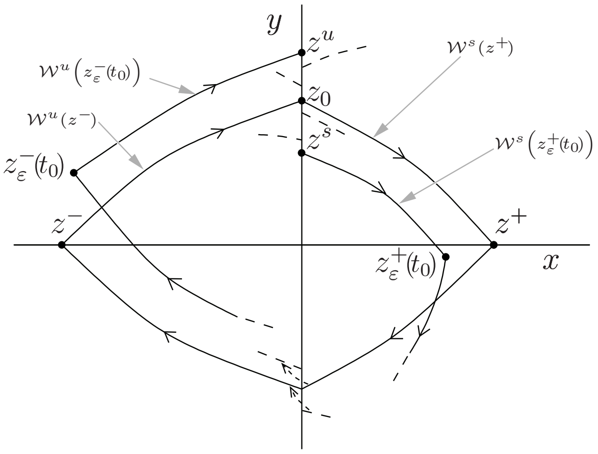


Figure 6. *Section of the unperturbed and perturbed invariant manifolds for $t = t_0$.*

Theorem 4.1. *Consider system (2.2), (2.18), and let $z_0 = W^s(z^+) \cap \Sigma$. Define the Melnikov function*

$$M(t_0) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left(\phi\left(t; t_0, z_0, 0\right), t\right) dt, \tag{4.2}$$

where

$$\phi(t; t_0, z_0, 0) = \begin{cases} \phi^-(t; t_0, z_0, 0) & \text{if } t \leq t_0, \\ \phi^+(t; t_0, z_0, 0) & \text{if } t > t_0 \end{cases} \tag{4.3}$$

is the piecewise-smooth heteroclinic orbit that exists for $r = 1$ and $\varepsilon = 0$. Assume that $M(t_0)$ possesses a simple zero at $\bar{t}_0$. Then the following holds.

(a) *If $r = 1$, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, one can find a simple zero $t_0^* = \bar{t}_0 + O(\varepsilon)$ of the function $\Delta(t_0, \varepsilon, 1)$. Hence, the curves $W^u(z_\varepsilon^-(t_0^*))$ and $W^s(z_\varepsilon^+(t_0^*))$ intersect transversally at some point, $z_h \in \Sigma$, ε-close to $z_0 \in \Sigma$, and*

$$\{\phi(t; t_0^*, z_h, \varepsilon),\ t \in \mathbb{R}\}$$

is a heteroclinic orbit between the periodic orbits Λ_ε^- and Λ_ε^+.

(b) *If $r < 1$, there exists ρ such that, given $\tilde{\varepsilon}, \tilde{r} > 0$ satisfying $0 < \frac{\tilde{r}}{\tilde{\varepsilon}} < \rho$, one can find δ_0 such that if $\varepsilon = \tilde{\varepsilon}\delta$ and $r = 1 - \tilde{r}\delta$, then, for $0 < \delta < \delta_0$, there exists a simple zero of the function $\Delta(t_0, \tilde{\varepsilon}\delta, 1 - \tilde{r}\delta)$ of the form $t_0^* = \bar{t}_0 + O(\frac{\tilde{r}}{\tilde{\varepsilon}}) + O(\delta)$. Hence, the curves $W^u(z_\varepsilon^-(t_0^*))$ and $W^s(z_\varepsilon^+(t_0^*))$ intersect Σ transversally at two points, $z_h^\pm \in \Sigma$, satisfying $z_h^+ = z_0 + O(\delta)$ and $z_h^- = z_h^+/r$, such that*

$$\left\{\phi(t; t_0^*, z_h^+, \tilde{\varepsilon}\delta, 1 - \tilde{r}\delta),\ t \in \mathbb{R}\right\}$$

is a heteroclinic orbit between the periodic orbits Λ_ε^- and Λ_ε^+.

Remark 4.1. Note that for $r = 1$, we recover the classical result given by the Melnikov method for heteroclinic orbits extended to the piecewise-smooth system (2.2).

Proof. Applying the fundamental theorem of calculus to the functions

$$s \longmapsto H_0^{+/-}\left(\phi^{+/-}\left(s; t_0, z^{s/u}, \varepsilon\right)\right),$$

we obtain

$$H_0^{+/-}\left(z^{s/u}\right) = H_0^{\pm}\left(\phi\left(T^{s/u}; t_0, z^{s/u}, \varepsilon\right)\right) + \int_{T^{s/u}}^{t_0} \frac{d}{ds} H_0^{+/-}\left(\phi^{+/-}\left(s; t_0, z^{s/u}, \varepsilon\right)\right) ds,$$

and then make $T^{s/u} = +/-\infty$. However, the limits

$$\lim_{t \to +/-\infty} \phi^{+/-}(t; t_0, z^{s/u}, \varepsilon)$$

do not exist because the flow at the respective stable/unstable manifolds tends to the periodic orbit $\Lambda_\varepsilon^{\pm}$. To avoid this limit, we proceed as follows.

Given t_0, we define

$$\begin{aligned} f_-(s) &= H_0^-\left(\phi^-\left(s; t_0, z^u, \varepsilon\right)\right) - H_0^-\left(\phi^-\left(s; t_0, z_\varepsilon^-(t_0), \varepsilon\right)\right), \quad s \le t_0, \\ f_+(s) &= H_0^+\left(\phi^+\left(s; t_0, z^s, \varepsilon\right)\right) - H_0^+\left(\phi^+\left(s; t_0, z_\varepsilon^+(t_0), \varepsilon\right)\right), \quad s \ge t_0, \end{aligned} \tag{4.4}$$

which are well-defined smooth functions because the flow is restricted to the stable and unstable invariant manifolds or to the hyperbolic periodic orbit and never crosses the switching manifold Σ.

Then we write (4.1) as

$$\Delta(t_0, \varepsilon, r) = r^2 f_-(t_0) - f_+(t_0) + r^2 H_0^-\left(z_\varepsilon^-(t_0)\right) - H_0^+\left(z_\varepsilon^+(t_0)\right). \tag{4.5}$$

Noting that

$$H_0^{\pm}(z_\varepsilon^{\pm}(t_0)) = \underbrace{H_0^{\pm}(z^{\pm})}_{c_1} + \varepsilon \underbrace{DH_0^{\pm}(z^{\pm})}_{\overset{\shortparallel}{0}} \frac{\partial z_\varepsilon^{\pm}(t_0)}{\partial \varepsilon}\Big|_{\varepsilon=0} + O(\varepsilon^2), \tag{4.6}$$

(4.5) becomes

$$\Delta(t_0, \varepsilon, r) = r^2 f_-(t_0) - f_+(t_0) + (r^2 - 1)c_1 + O(\varepsilon^2). \tag{4.7}$$

We apply the fundamental theorem of calculus to the functions (4.4) to compute

$$\begin{aligned} f_-(t_0) &= f_-(T^u) + \int_{T^u}^{t_0} f_-'(s) ds = f_-(T^u) + \varepsilon \int_{T^u}^{t_0} \Big(\left\{H_0^-, H_1^-\right\}\left(\phi^-\left(s; t_0, z^u, \varepsilon\right), s\right) \\ &\quad - \left\{H_0^-, H_1^-\right\}\left(\phi^-\left(s; t_0, z_\varepsilon^-(t_0), \varepsilon\right), s\right)\Big) ds, \\ f_+(t_0) &= f_+(T^s) - \int_{t_0}^{T^s} f_+'(s) ds = f_+(T^s) - \varepsilon \int_{t_0}^{T^s} \Big(\left\{H_0^+, H_1^+\right\}\left(\phi^+\left(s; t_0, z^s, \varepsilon\right), s\right) \\ &\quad - \left\{H_0^+, H_1^+\right\}\left(\phi^+\left(s; t_0, z_\varepsilon^+(t_0), \varepsilon\right), s\right)\Big) ds. \end{aligned} \tag{4.8}$$

Due to the hyperbolicity of the periodic orbits $\Lambda_\varepsilon^\pm$, the flow on $W^{s/u}(\Lambda_\varepsilon^{+/-})$ converges exponentially to them (forward or backward in time). That is, there exist positive constants C, λ, and s_0 such that

$$\left|\phi^+\left(s;t_0,z^s,\varepsilon\right)-\phi^+\left(s;t_0,z_\varepsilon^+(t_0),\varepsilon\right)\right|<Ce^{-\lambda s}\quad\forall s>s_0,$$

and similarly for ϕ^-. This allows one to make $T^{s/u}\to +/-\infty$ in equations (4.8), since

$$\lim_{s\to\pm\infty}f_\pm(s)=0,$$

and, moreover, the improper integrals converge in the limit.

Now, expanding the expressions in (4.8) in powers of ε, we find

$$\begin{aligned} f_-(t_0)&=\varepsilon\int_{-\infty}^{t_0}\left\{H_0^-,H_1^-\right\}\left(\phi^-\left(s;t_0,z_0,0\right),s\right)ds+O(\varepsilon^2),\\ f_+(t_0)&=-\varepsilon\int_{t_0}^{\infty}\left\{H_0^+,H_1^+\right\}\left(\phi^+\left(s;t_0,z_0,0\right),s\right)ds+O(\varepsilon^2),\end{aligned}\tag{4.9}$$

where we have used property (4.6) to include the second terms in the integrals into the higher order terms. Finally, substituting (4.9) into (4.7), we obtain

$$\Delta(t_0,\varepsilon,r)=(r^2-1)c_1+\varepsilon M(t_0)+O(\varepsilon^2)+O(\varepsilon\,(r-1)),\tag{4.10}$$

where $M(t_0)$ is as defined in (4.2).

We now distinguish between the cases $r=1$ and $r<1$. If $r=1$, we recover the classical expression for the distance between the perturbed invariant manifolds. By applying the implicit function theorem, it is easy to show that if $M(t_0)$ has a simple zero at $\bar t_0$, then $\Delta(t_0,\varepsilon,1)$ has a simple zero at $t_0^*=\bar t_0+O(\varepsilon)$. Thus, the curves $W^u(z_\varepsilon^-(t_0^*))$ and $W^s(z_\varepsilon^+(t_0^*))$ intersect Σ transversally at some point $z_h=z^u(t_0^*)=z^s(t_0^*)\in\Sigma$, ε-close to $z_0\in\Sigma$. Therefore,

$$\{\phi(t;t_0^*,z_h,\varepsilon),\,t\in\mathbb{R}\}$$

is a heteroclinic orbit between the periodic orbits Λ_ε^- and Λ_ε^+.

If $r<1$, we define $\varepsilon=\tilde\varepsilon\delta$ and $r=1-\tilde r\delta$, and (4.10) becomes

$$\frac{\Delta(t_0,\tilde\varepsilon\delta,1-\tilde r\delta)}{\delta}=-2\tilde r c_1+\tilde\varepsilon M(t_0)+O(\delta).\tag{4.11}$$

Then we argue as in Theorem 3.3. As $M(t_0)$ is a smooth periodic function, it possesses at least one local maximum. Let t_M be the closest value to $\bar t_0$ where $M(t_0)$ possesses a local maximum, and assume $M'(t_0)\neq 0$ for all t_0 between $\bar t_0$ and t_M. If $M'(t_0)$ vanishes between $\bar t_0$ and t_M, we then take t_M to be the closest value to $\bar t_0$ such that $M'(t_0)=0$ to ensure that $M'(t_0)\neq 0$ between $\bar t_0$ and t_M. We then define $\rho:=\frac{M(t_M)}{2c_1}$. Then if

$$0<\frac{\tilde r}{\tilde\varepsilon}<\rho,$$

there exists $\widehat{t}_0$ $\frac{\tilde{r}}{\tilde{\varepsilon}}$-close to $\bar{t}_0$ such that

$$-2\tilde{r}c_1 + M(\widehat{t}_0) = 0$$

and $M'(\widehat{t}_0) \neq 0$. Hence, we can apply the implicit function theorem to (4.11) near the point $(t_0, \delta) = (\widehat{t}_0, 0)$ to conclude that there exists δ_0 such that if $0 < \delta < \delta_0$, then one can find

$$t_0^* = \widehat{t}_0 + O(\delta) = \bar{t}_0 + O(\delta) + O(\tilde{r}/\tilde{\varepsilon}),$$

which is a simple solution of (4.11).

Hence, arguing similarly as for $r = 1$, there exist two points $z_h^+ = z^s(t_0^*) = z_0 + O(\delta)$ and $z_h^- = z^u(t_0^*) = z_0/r + O(\delta)r$ such that $z_h^+ = r z_h^-$ and

$$\left\{\phi(t; t_0^*, z_h^+, \tilde{\varepsilon}\delta, 1 - \tilde{r}\delta),\, t \in \mathbb{R}\right\},$$

where

$$\phi\left(t; t_0^*, z_h^+, \varepsilon, r\right) = \begin{cases} \phi^-\left(t; t_0, t_0^*, z_h^+/r, \varepsilon\right) & \text{if } t \leq t_0^*, \\ \phi^+\left(t; t_0, t_0^*, z_h^+, \varepsilon\right) & \text{if } t \geq t_0^* \end{cases}$$

is a heteroclinic orbit between the periodic orbits Λ_ε^- and Λ_ε^+. ■

5. Example: The rocking block.

5.1. System equations. In order to illustrate our results, we consider the mechanical system shown in Figure 7, which consists of a rocking block under a horizontal periodic forcing given by

$$a_H(t) = \varepsilon\alpha g \cos\left(\Omega t + \theta\right). \tag{5.1}$$

This system was first studied in [Hou63]. The fixed angle between one side of the block and the diagonal through the mass center is denoted by α. When there is rotation, the angular displacement from the vertical is given by αx. Then the equations that govern the motion of the block, after a time scaling, are given by

$$\alpha\ddot{x} + \operatorname{sign}(x)\sin(\alpha(1 - \operatorname{sign}(x)\, x)) = -\alpha\varepsilon\cos\left(\alpha\left(1 - \operatorname{sign}(x)\, x\right)\right)\cos\left(\omega t\right), \tag{5.2}$$

$$\dot{x}(t_A^+) = r\dot{x}(t_A^-) \qquad (x = 0), \tag{5.3}$$

where $\omega = \sqrt{\frac{4R}{3g}}\Omega$ (see, for example, [YCP80, SK84, Hog89] for details).

The last equation, (5.3), simulates the loss of energy of the block at every impact with the ground, as described in section 2.3, and the function

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0 \end{cases} \tag{5.4}$$

distinguishes between the two modes of movement: rocking about the point O when $x > 0$ or rocking about O' when $x < 0$. Hence (5.2)–(5.4) are piecewise-smooth, conditions C.1–C.5 are satisfied, and so results from previous sections can be applied. However, as our purpose

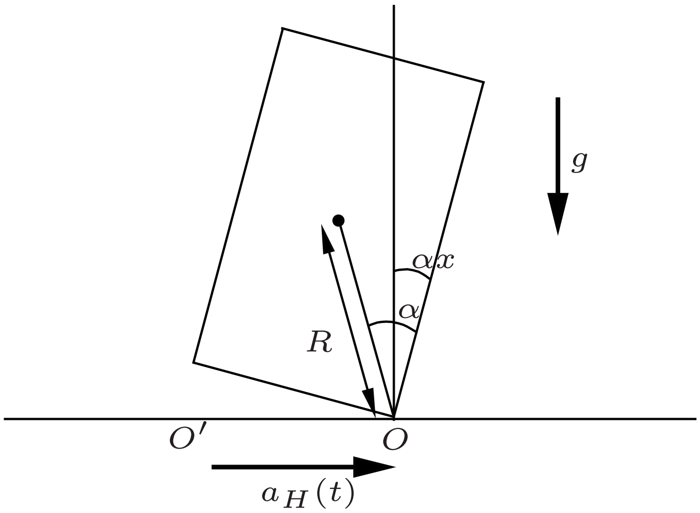


Figure 7. *Rocking block.*

is to compare with [Hog89], we will consider the terms linear in α of (5.2) instead, which will permit us to perform explicit analytical computations. This linearization is equivalent to the assumption that the block is slender [Hog89]. Thus, the system that we will consider, written in the form of (2.2), is

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - 1 - \varepsilon\cos(\omega t)\end{aligned}\right\} \text{ if } x > 0, \tag{5.5}$$

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= x + 1 - \varepsilon\cos(\omega t)\end{aligned}\right\} \text{ if } x < 0, \tag{5.6}$$

$$y(t_A^+) = r y(t_A^-) \qquad (x = 0), \tag{5.7}$$

where the perturbation becomes a smooth function due to the linearization.

If $r = 1$, system (5.5)–(5.6) can be written in the form (2.5) using the Hamiltonian function

$$H_\varepsilon(x, y, t) = H_0(x, y) + \varepsilon H_1(x, t), \tag{5.8}$$

where

$$H_0(x, y) = \begin{cases} \dfrac{y^2}{2} - \dfrac{x^2}{2} + x & \text{if } x > 0,\\ \dfrac{y^2}{2} - \dfrac{x^2}{2} - x & \text{if } x < 0,\end{cases} \tag{5.9}$$

and

$$H_1(x, t) = x\cos(\omega t) \tag{5.10}$$

is a C^∞ and T-periodic function, with $T = 2\pi/\omega$.

In addition, when $\varepsilon = 0$, conditions C.1–C.5 are fulfilled, and the phase portrait for the system (5.5)–(5.6) is equivalent to the one shown in Figure 1. That is, it possesses an invisible fold-fold of center type at the origin and two saddle points at $(1, 0)$ and $(-1, 0)$ connected by two heteroclinic orbits.

Furthermore, the origin is surrounded by a continuum of periodic orbits whose periods monotonically increase as they approach the heteroclinic connections. This can be shown as follows. Using (2.15) and (2.16), the symmetries of the Hamiltonian (5.9) and assuming $y_0 > 0$, these periods are given by

$$\begin{aligned}\alpha(y_0) &= 4\int_0^{1-\sqrt{1-y_0^2}} \frac{1}{\sqrt{y_0^2 + x^2 - 2x}}dx \\ &= 2\ln\left(\frac{1+y_0}{1-y_0}\right),\end{aligned} \tag{5.11}$$

and hence $\alpha'(y_0) > 0$.

5.2. Existence of periodic orbits. We first study the persistence of (n, m)-periodic orbits for $r = 1$ in (5.7) by applying Theorem 3.1. The subharmonic Melnikov function (3.3) becomes

$$M^{n,m}(t_0) = -\int_0^{nT} \Pi_y(q_c(t))\cos(\omega\,(t+t_0))dt, \tag{5.12}$$

where $q_c(t)$ is the periodic orbit of the unperturbed version of system (5.5)–(5.6) with energy level $c = \frac{\bar{y}_0^2}{2}$ satisfying $q_c(0) = (0, \bar{y}_0)$ and

$$\bar{y}_0 = \alpha^{-1}\left(\frac{nT}{m}\right) = \frac{e^{\frac{nT}{2m}} - 1}{e^{\frac{nT}{2m}} + 1}. \tag{5.13}$$

We now want to obtain an explicit expression for (5.12). Thus we first note that the solution of system (5.5)–(5.6) with initial condition (x_0, y_0) at $t = t_0$ is given by

$$x^\pm(t) = C_1^\pm e^t + C_2^\pm e^{-t} \pm 1, \tag{5.14}$$

$$y^\pm(t) = C_1^\pm e^t - C_2^\pm e^{-t}, \tag{5.15}$$

where

$$C_1^\pm = \frac{x_0 + y_0 \mp 1}{2}e^{-t_0},\ C_2^\pm = \frac{x_0 - y_0 \mp 1}{2}e^{t_0}. \tag{5.16}$$

As explained in section 2.4, the superscript + is applied if $x_0 > 0$ or $x_0 = 0$ and $y_0 > 0$, and the superscript $-$ otherwise.

Assuming $x_0 = 0$ and $y_0 = \bar{y}_0 > 0$, it can be shown that

$$\Pi_y(q_c(t)) = \begin{cases} C_1e^t - C_2e^{-t} & \text{if } 0 \le t \le \dfrac{nT}{2m}, \\ -C_1e^{t-\frac{nT}{2m}} + C_2e^{-t+\frac{nT}{2m}} & \text{if } \dfrac{nT}{2m} \le t \le \dfrac{nT}{m}, \end{cases} \tag{5.17}$$

where

$$C_1 = \frac{\bar{y}_0 - 1}{2}, \; C_2 = \frac{-\bar{y}_0 - 1}{2}. \tag{5.18}$$

Thus, (5.12) becomes

$$M^{n,m}(t_0) = -\sum_{j=0}^{m-1} \left(\int_0^{\frac{nT}{2m}} \left(C_1 e^t - C_2 e^{-t}\right) \cos\left(\omega\left(t + t_0 + j\frac{nT}{m}\right)\right) dt \right.$$
$$\left. + \int_{\frac{nT}{2m}}^{\frac{nT}{m}} \left(-C_1 e^{t-\frac{nT}{2m}} + C_2 e^{-t+\frac{nT}{2m}}\right) \cos\left(\omega\left(t + t_0 + j\frac{nT}{m}\right)\right) dt \right),$$

and, after some computations, we have

$$M^{n,m}(t_0) = \begin{cases} -\dfrac{4}{\omega^2+1}\cos(\omega t_0) & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases} \tag{5.19}$$

As $M^{n,1}(t_0)$ has two simple zeros, $\bar{t}_0^1 = \frac{T}{4}$ and $\bar{t}_0^2 = \frac{3T}{4}$, by Theorem 3.1, if $\varepsilon > 0$ is small enough, the nonautonomous system (5.5)–(5.6) possesses two subharmonic $(n, 1)$-periodic orbits. In addition, the initial conditions of these periodic orbits are ε-close to

$$(0, \bar{y}_0, \bar{t}_0^1) = \left(0, \frac{e^{\frac{nT}{2}} - 1}{e^{\frac{nT}{2}} + 1}, \frac{T}{4}\right) \tag{5.20}$$

and

$$(0, \bar{y}_0, \bar{t}_0^2) = \left(0, \frac{e^{\frac{nT}{2}} - 1}{e^{\frac{nT}{2}} + 1}, \frac{3T}{4}\right), \tag{5.21}$$

respectively.

Proceeding as in Remark 3.2, one can solve (3.2) numerically with $m = 1$ and find the initial conditions for such a periodic orbit. In Figure 8 we show the results for $n = 5$. Both periodic orbits are obtained by using the points given in (5.20) and (5.21) to initiate Newton's method. Then, following the solution, ε was increased up to $\varepsilon = 1.6565 \cdot 10^{-2}$.

Since the subharmonic Melnikov function is identically zero, nothing can be said about the existence of (n, m)-periodic orbits with $m > 1$ (ultrasubharmonic orbits), using the first order analysis given in this paper.

However, if instead of (5.10) one considers the perturbation

$$H_1(x, t) = x\left(\cos(\omega t) + \cos(k\omega t)\right),$$

then it can be seen that the corresponding Melnikov function possesses simple zeros for $m = k$ and n relatively prime odd integers. Thus, periodic orbits impacting $m > 1$ times with the switching manifold can exist if higher harmonics of the perturbation are considered.

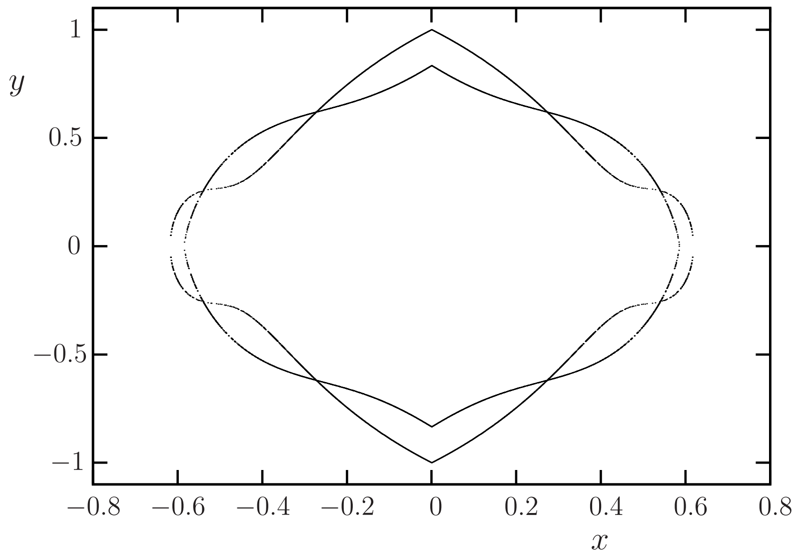


Figure 8. *Periodic orbits for $n = 5$ and $m = 1$, $r = 1$, $\omega = 5$, and $\varepsilon = 1.6565 \cdot 10^{-2}$. Their initial conditions are ε-close to the points given in (5.20) and (5.21).*

Let us now introduce the energy dissipation described in section 3.2 and consider the whole system (5.5)–(5.7) with $r < 1$ using the Hamiltonian perturbation (5.8). From Theorem 3.3, simple zeros of the Melnikov function (5.19) also guarantee the existence of $(n, 1)$-periodic orbits when $1 - r$ is small enough compared to ε. More precisely, taking

$$\varepsilon = \tilde{\varepsilon}\delta,\ r = 1 - \tilde{r}\delta, \tag{5.22}$$

condition (3.14) becomes

$$0 < \frac{\tilde{r}}{\tilde{\varepsilon}} < \frac{1}{2}\left(\frac{e^{\frac{nT}{2}} + 1}{e^{\frac{nT}{2}} - 1}\right)^2 M^{n,1}(t_M) := \rho, \tag{5.23}$$

where $M^{n,1}(t_M) = M^{n,1}(\frac{T}{2}) = \frac{4}{\omega^2+1}$ is the maximum value of the Melnikov function (5.19). Then, according to Theorem 3.3, there exists an $(n, 1)$-periodic orbit if $\delta > 0$ is small enough. The initial condition of the periodic orbit is located in a δ-neighborhood of the point $(x_0, y_0, t_0) = (0, \bar{y}_0, \widehat{t}_0)$, where $\bar{y}_0$ is defined in (5.13), such that

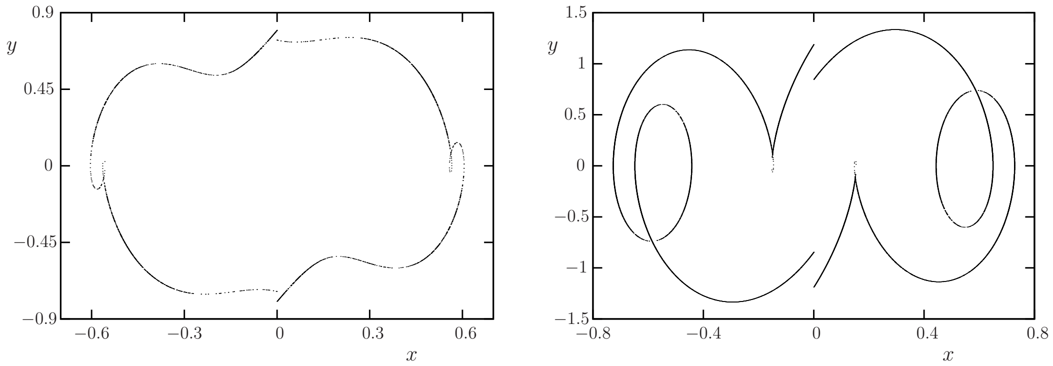
$$\alpha(\bar{y}_0) = nT$$

and $\widehat{t}_0$ is given by the simple zeros of (3.13), which becomes

$$-2\tilde{r}\bar{y}_0^2 + \tilde{\varepsilon}M^{n,1}(t_0) = 0. \tag{5.24}$$

Hence we find

$$\widehat{t}_0^i = \frac{1}{\omega}\arccos\left(-\frac{\omega^2+1}{2}\left(\frac{e^{\frac{nT}{2}} - 1}{e^{\frac{nT}{2}} + 1}\right)^2 \frac{\tilde{r}}{\tilde{\varepsilon}}\right) + (i-1)\frac{T}{2},\ i = 1, 2. \tag{5.25}$$



(a) $\delta = 1.04$ (b) $\delta = 4.125$

Figure 9. $(5,1)$*-periodic orbits for* $\omega = 5$ *and* $\frac{\tilde{r}}{\tilde{\varepsilon}} = 0.07$*. Following the obtained solution, the perturbation parameter* δ *has been increased up to its maximum value. Initial conditions close to* $(\bar{y}_0, \widehat{t}_0^1)$ *and* $(\bar{y}_0, \widehat{t}_0^2)$ *have been used in* (a) *and* (b)*, respectively.*

As before, if we set $n = 5$ and $\omega = 5$, then expression (5.23) becomes

$$0 < \frac{\tilde{r}}{\tilde{\varepsilon}} < 0.0914. \tag{5.26}$$

Hence, for any fixed ratio $\frac{\tilde{r}}{\tilde{\varepsilon}}$ satisfying (5.26) there exist two points, $(\bar{y}_0, \widehat{t}_0^i)$, $i = 1, 2$, such that if δ is small enough, (3.10) possesses a solution δ-close to them. Such a solution is an initial condition for an $(n,1)$-periodic orbit of system (5.5)–(5.7), with $r = 1 - \tilde{r}\delta$ and $\varepsilon = \tilde{\varepsilon}\delta$.

In Figure 9 some of these orbits are shown for one value of the ratio $\frac{\tilde{r}}{\tilde{\varepsilon}}$ satisfying (5.26). Two different periodic orbits are shown whose initial conditions are δ-close to $(\bar{y}_0, \widehat{t}_0^1)$ and $(\bar{y}_0, \widehat{t}_0^2)$. In both cases, δ tracks the solution, up to values where solutions of (3.9) can no longer be found. These values are used in the simulations shown in Figure 9. Note that, above the limiting value of the ratio given in (5.26), no $(5,1)$-periodic orbits were found for $\omega = 5$ for any value of δ.

5.3. Existence curves. We now derive existence curves for the $(n,1)$-periodic orbits (n odd) and compare them with results obtained in [Hog89].

By integrating the linearized system and imposing symmetry conditions on the orbit, an explicit expression for the existence of $(n,1)$-periodic orbits in the r-ε plane was obtained in [Hog89], namely,

$$\varepsilon_{\min}(R) = \frac{\left(1+\omega^2\right) R \left(1 - \cosh\left(\frac{nT}{2}\right)\right)}{\sqrt{\omega^2 \sinh^2\left(\frac{nT}{2}\right) R^2 + (2-R)^2 \left(1 + \cosh\left(\frac{nT}{2}\right)\right)^2}}, \tag{5.27}$$

where $R = 1 - r$. This exact global formula provides, for every n and r, the minimum value of the amplitude of the perturbation $\varepsilon_{\min}$ such that an $(n,1)$-periodic orbit exists.

Unlike in [Hog89], we obtain similar curves in the r-ε plane by applying Theorem 3.3. As the existence of such orbits in Theorem 3.3 is proven using the implicit function theorem, the existence of these orbits is valid only locally. Hence, such existence curves are obtained

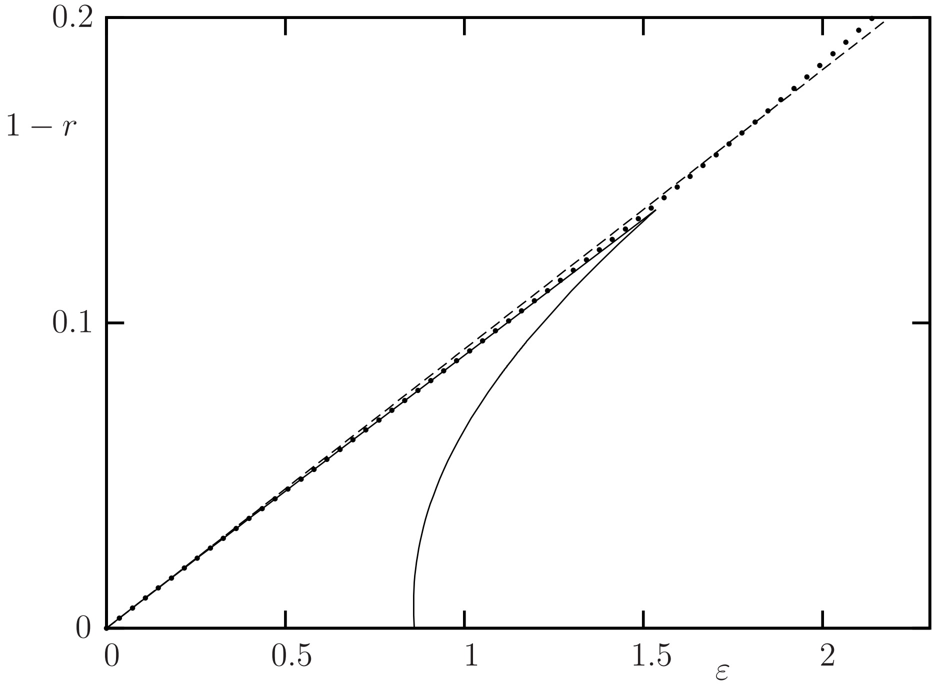


Figure 10. *Existence curves of a* $(5,1)$*-periodic orbit for* $\omega = 5$*. Expression derived from Theorem* 3.3 *(solid line), expression for* ε_{min} *derived from* [Hog89] *(dotted line), and* $1-r=\rho\varepsilon$ *(dashed line).*

by numerical continuation of the local periodic orbits. However, our method is more general; it does not depend on the details of the system, the type of perturbation, or any symmetry assumptions. Moreover, it can also be applied when considering ultrasubharmonic periodic orbits ($m > 1$) as long as the hypothesis of Theorem 3.3 are fulfilled.

The limiting condition provided by Theorem 3.3 is given in (5.23). Thus, for a given r close to 1 (that is, $\tilde{r}\delta$ close to 0), it is natural to fix $\tilde{r}$ and minimize ε by maximizing the ratio in (5.23), setting $\frac{1-r}{\varepsilon} = \frac{\tilde{r}\delta}{\tilde{\varepsilon}\delta} = \frac{\tilde{r}}{\tilde{\varepsilon}} = \rho$. As can be seen in Figure 10, this curve, which is the straight line

$$1 - r = \rho\varepsilon,$$

is very close to the one obtained in [Hog89], given by (5.27).

One might think that these two curves should be identical due to the fact that the system appears to be linear in ε and piecewise-linear in x. Nevertheless, it is important to emphasize that the system solutions are in fact nonlinear in ε, since it is necessary to solve a nonlinear equation for the time when the solution crosses the switching manifold. Therefore, the curve $\frac{\tilde{r}}{\tilde{\varepsilon}} = \rho$, obtained using Melnikov theory, is not identical to the one obtained in [Hog89], but both curves are very close when ε is small (see Figure 10).

As our method applies to general systems, the existence of periodic orbits given by Theorem 3.3 is only valid for $\delta < \delta_0 = \delta_0(\frac{\tilde{r}}{\tilde{\varepsilon}})$. Moreover, δ_0 tends to zero as $\frac{\tilde{r}}{\tilde{\varepsilon}} \to \rho$, as it is derived from the implicit function theorem. This is because, as $\frac{\tilde{r}}{\tilde{\varepsilon}} \to \rho$, the solution $\hat{t}_0$ which solves (5.24) tends to t_M, where t_M is a maximum of $M^{n,1}$ and hence $(M^{n,1})'(t_M) = 0$. Therefore, as $\hat{t}_0$ approaches t_M, the domain of validity provided by the implicit function theorem tends to zero, and thus so does δ_0 ($\delta_0 = O((M^{n,1})'(\hat{t}_0))$). As a consequence, it is not possible to find $\delta^* = \delta_0(\rho) > 0$ such that for any $\delta < \delta^*$ we could apply Theorem 3.3 to obtain periodic orbits. Hence, the condition $\frac{\tilde{r}}{\tilde{\varepsilon}} = \rho$ cannot be used to derive a limiting relation between r and ε if we use a first order perturbation theory as in the Melnikov approach. Instead, we proceed as follows.

We first fix n odd and $\omega > 0$. Then, for every ratio $0 < \frac{\tilde{r}}{\tilde{\varepsilon}} < \rho$, we increase δ from 0 to δ_0 by numerically following the solution obtained using as initial condition one of the values provided in (5.20) or (5.21). This results in a curve in the r-ε plane parametrized by the ratio $\frac{\tilde{r}}{\tilde{\varepsilon}}$.

As our result is only locally valid, in order to compare it with [Hog89] we have to check whether both curves are tangent at the origin. From (5.27) we easily obtain

$$\varepsilon'_{\min}(0) = -\frac{1+\omega^2}{2}\left(\frac{e^{\frac{nT}{2}}-1}{e^{\frac{nT}{2}}+1}\right)^2 = -\frac{1}{\rho},$$

which, by the inverse function theorem, tells us that both curves are tangent at the origin.

In Figure 10, we show an example for $n = 5$ and $\omega = 5$ using initial conditions near (5.20). As can be seen, the expression derived from Theorem 3.3 (solid line) provides, for every value of r, both the maximum and minimum values of ε for which a $(5, 1)$-periodic orbit exists, according to our method. The lower boundary derived in [Hog89], $(\varepsilon_{\min}(\cdot))^{-1}(\varepsilon)$ is also shown (dotted line). As demonstrated above, both curves are tangent at the origin, with slope equal to ρ. Note, however, that the minimum value does not coincide with the line $1 - r = \rho\varepsilon$, although their difference tends to zero as $r \to 1$. This confirms that one cannot derive the minimum value of ε from condition (5.23) for every fixed r.

6. Conclusions.

In this paper we have extended the classical Melnikov methods for subharmonic orbits and homoclinic/heteroclinic connections to piecewise-defined Hamiltonian systems with a piecewise-defined periodic Hamiltonian perturbation. We rigorously prove that when the unperturbed system has a piecewise-continuous Hamiltonian, the classical method also holds. In this case, simple zeros of the modified classical and subharmonic Melnikov functions guarantee the existence of subharmonic orbits and heteroclinic connections, respectively, in the perturbed system. We have also considered the case when the solution trajectories are discontinuous at the switching manifold, as in the case of an impacting system with energy loss represented by a coefficient of restitution. In this case, the unperturbed system has the origin as a global attractor. We have shown that the same results hold when this restitution coefficient is small enough with respect to the amplitude of the periodic perturbation.

In addition, our method provides a constructive way to find initial conditions for subharmonic orbits. In this way, we have found periodic orbits in the rocking block problem [Hog89]. We have also numerically obtained existence curves for these periodic orbits, and we have compared them with those given in [Hog89].

Future work should consider an extension of the method to quasi-periodic or almost-periodic perturbations. In [MS89] the authors present a generalization of the Melnikov method for this class of perturbation to smooth systems to show that the perturbed system has homoclinic trajectories. The Poincaré stroboscopic map is not a suitable tool for the systems treated in [MS89], as is the case for the periodic perturbations for piecewise-smooth systems that we have considered here. Hence, we believe that the results presented here can also be extended to quasi-periodic perturbations by the use of the impact map.

REFERENCES

[BF08] F. Battelli and M. Fečkan, *Homoclinic trajectories in discontinuous systems*, J. Dynam. Differential Equations, 20 (2008), pp. 337–376.

[BF11] F. Battelli and M. Fečkan, *Nonsmooth homoclinic orbits, Melnikov functions and chaos in discontinuous systems*, Phys. D, 2011; available online from http://dx.doi.org/10.1016/j.physd.2011.05.018.

[BK91] B. Bruhn and B. P. Koch, *Heteroclinic bifurcations and invariant manifolds in rocking block dynamics*, Z. Naturforsch. A, 46 (1991), pp. 481–490.

[CFGF11] V. Carmona, S. Fernández-García, and E. Freire, *Periodic orbits for perturbations of piecewise linear systems*, J. Differential Equations, 250 (2011), pp. 2244–2266.

[DL12] Z. Du and Y. Li, *Bifurcation of periodic orbits with multiple crossings in a class of planar Filippov systems*, Math. Comput. Modelling, 55 (2012), pp. 1072–1082.

[DLZ08] Z. Du, Y. Li, and W. Zhang, *Bifurcation of periodic orbits in a class of planar Filippov systems*, Nonlinear Anal., 69 (2008), pp. 3610–3628.

[DZ05] Z. Du and W. Zhang, *Melnikov method for homoclinic bifurcation in nonlinear impact oscillators*, Comput. Math. Appl., 50 (2005), pp. 445–458.

[GH83] J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, 4th ed., Appl. Math. Sci. 42, Springer, New York, 1983.

[GST11] M. Guardia, T. M. Seara, and M. A. Teixeira, *Generic bifurcations of low codimension of planar Filippov systems*, J. Differential Equations, 250 (2011), pp. 1967–2023.

[Hog89] S. J. Hogan, *On the dynamics of rigid block motion under harmonic forcing*, Proc. Roy. Soc. Lond. Ser. A, 425 (1989), pp. 441–476.

[Hog92] S. J. Hogan, *Heteroclinic bifurcations in damped rigid block motion*, Proc. Roy. Soc. Lond. Ser. A, 439 (1992), pp. 155–162.

[Hou63] G. W. Housner, *The behaviour of inverted pendulum structures during earthquakes*, Bull. Seism. Soc. Amer., 53 (1963), pp. 403–417.

[KKY97] M. Kunze, T. Küpper, and J. You, *On the application of KAM theory to discontinuous dynamical systems*, J. Differential Equations, 139 (1997), pp. 1–21.

[Kuk07] P. Kukučka, *Melnikov method for discontinuous planar systems*, Nonlinear Anal., 66 (2007), pp. 2698–2719.

[Kun00] M. Kunze, *Non-smooth Dynamical Systems*, Springer, New York, 2000.

[LH10] X. Liu and M. Han, *Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 20 (2010), pp. 1379–1390.

[MS89] K. Meyer and G. Sell, *Melnikov transforms, Bernoulli bundles and almost periodic perturbations*, Trans. Amer. Math. Soc., 314 (1989), pp. 63–105.

[SK84] P. D. Spanos and A.-S. Koh, *Rocking of rigid blocks due to harmonic shaking*, J. Eng. Mech. ASCE, 110 (1984), pp. 1627–1642.

[TA07] G. Tigan and A. Astolfi, *A note on a piecewise-linear Duffing-type system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 17 (2007), pp. 4425–4429.

[XFR09] W. Xu, J. Feng, and H. Rong, *Melnikov's method for a general nonlinear vibro-impact oscillator*, Nonlinear Anal., 71 (2009), pp. 418–426.

[Yag] K. Yagasaki, *Application of the subharmonic Melnikov method to piecewise-smooth systems*, submitted for publication, preprint available at http://www.eng.niigata-u.ac.jp/~yagasaki/preprints/p09h.pdf.

[YCP80] C.-S. Yim, A. K. Chopra, and J. Penzien, *Rocking response of rigid blocks to earthquakes*, Earthquake Engrg. Struct. Dyn., 8 (1980), pp. 565–587.